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ELEMENTS OF ALGEBRA
GEOMETRICAL DRAWING
THE TRANSITION SPIRAL
EARTHWORK

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PREFACE

The volumes of the International Library of Technology are made up of Instruction Papers, or Sections, comprising the various courses of instruction for students of the International Correspondence Schools. The original manuscripts are prepared by persons thoroughly qualified both technically and by experience to write with authority, and in many cases they are regularly employed elsewhere in practical work as experts. The manuscripts are then carefully edited to make them suitable for correspondence instruction. The Instruction Papers are written clearly and in the simplest language possible, so as to make them readily understood by all students. Necessary technical expressions are clearly explained when introduced.

The great majority of our students wish to prepare themselves for advancement in their vocations or to qualify for more congenial occupations. Usually they are employed and able to devote only a few hours a day to study. Therefore every effort must be made to give them practical and accurate information in clear and concise form and to make this information include all of the essentials but none of the non-essentials. To make the text clear, illustrations are used freely. These illustrations are especially made by our own Illustrating Department in order to adapt them fully to the requirements of the text.

In the table of contents that immediately follows are given the titles of the Sections included in this volume, and under each title are listed the main topics discussed. At the end of the volume will be found a complete index, so that any subject treated can be quickly found.

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CONTENTS

ELEMENTS OF ALGEBRA

(PART 1)

FUNDAMENTAL OPERATIONS

USE OF LETTERS

1. In arithmetic, numbers are represented by the figures 1, 2, 3, 4, etc. There is no reason, however, why numbers may not be represented by other symbols, such as letters, if rules are provided for their use.

In algebra, numbers are represented by both figures and letters. It will be seen later that the use of letters often simplifies the solution of examples and shortens calculations.

The principal advantage of letters is that they are general in their meaning. Thus, unlike figures, the letter a does not stand for the number 1, the letter b for 2, c for 3, etc., but *any* letter may be taken to represent *any* number, it being only necessary that a letter shall always stand for the same number *in the same example.*

2. To illustrate this difference between letters and figures, consider an example, as follows: If a farmer exchanges 20 bushels of oats, worth 40 cents per bushel, for 8 bushels of wheat, what is the price of the wheat per bushel? Working this example by arithmetic, it is necessary to find first the value of the oats, which is 20×40 cents $= 800$ cents, and then divide this result by 8 to find the

price per bushel of the wheat, obtaining 800 cents $\div$ 8 $=$ 100 cents. Any other similar example would be worked in the same manner. If, however, letters are used instead of figures, the final expression will be a formula (which, when expressed in words, becomes a rule), which can be applied to *any* example of this kind by substituting for the letters the numerical values assigned to them in the particular example under consideration. Thus,

Let $a =$ number of bushels of oats;
$b =$ price per bushel of oats;
$c =$ number of bushels of wheat;
$d =$ price per bushel of wheat.

The price of the oats is then $a \times b$; dividing this product by c, the price of the wheat per bushel is $\frac{a \times b}{c}$, which is denoted by d. Therefore, d equals $\frac{a \times b}{c}$, or

$$d = \frac{a \times b}{c}$$

This last expression is called an *equation* in algebra, but when used to solve examples like the foregoing, it is called a *formula*. As given here, this formula is perfectly general; a may represent any number of bushels of oats; b, any price of oats per bushel; c, any number of bushels of wheat; and d, the resulting price per bushel of wheat. Expressing the formula in words,

The price paid per bushel of wheat is equal to the number of bushels of oats multiplied by the price of oats per bushel, and the product divided by the number of bushels of wheat received in exchange.

The words in Italics constitute a general rule and apply to any similar exchange of any two commodities, by merely changing the words *oats* and *wheat* to whatever else is bartered, and *bushels* to whatever other units of measure are used.

3. Standing by itself the equation $d = \frac{a \times b}{c}$ has practically no meaning, except as indicating that certain operations are to be performed. When meanings are given to the letters, the equation becomes intelligible at once, and when numerical values are assigned to the quantities represented by the letters a, b, and c, the value of d can be determined. For example, if a and c represent the number of bushels of oats and wheat, respectively, involved in any exchange and b and d their respective prices per bushel, then from the equation (formula) it is seen at once that the price of the wheat is to be found by multiplying the number of bushels of oats by the price per bushel and dividing the product by the number of bushels of wheat exchanged In other words, it is known just what operations are required to find the price per bushel of the wheat. If, further, the number of bushels of oats involved in the transaction is 20 and of wheat is 8, and the price of the oats is 40 cents per bushel, then these values are substituted for the quantities the letters represent, thus,

$$d = \frac{20 \times 40}{8} = 100 \text{ cents}$$

4. The foregoing is a very simple example; it has been introduced merely to give some idea of what algebra is used for and a reason for studying it. The conditions involved in any particular problem require that the quantities involved be subjected to various operations, arrangements, and combinations, in order that the final expression may be reduced to as simple a form as possible. The operations are practically the same as in arithmetic, only more general; viz., addition, subtraction, multiplication, division, involution, evolution, and factoring. The operation of factoring is a particularly important one in algebra, as will be pointed out later.

Many practical problems can be solved by algebra that are incapable of solution by arithmetic, and many others are readily solved that can be solved only with great difficulty by arithmetic.

5. An **equation** is a statement of equality between two expressions. Thus, $x + y = 8$ is an equation, and means that the sum of the numbers represented by x and y is equal to 8. Examples are solved in algebra by the aid of equations, in which numbers are represented both by letters and by figures. The following simple example will give an idea of the method of solution:

EXAMPLE —If an iron rail 80 feet long is cut in two so that one part is four times as long as the other, how long is the shorter part?

SOLUTION —Any letter may represent any number, therefore:

Let $\quad x =$ length of shorter part

Then, $4 \times x$ (written $4x$) $=$ length of longer part

But the sum of the two parts must equal the total length, 80 ft.

Hence, $\quad x + 4x = 80$

Adding x and $4x$, $\quad 5x = 80$

Whence, dividing by 5, $\quad x = 6$ ft. Ans.

6. The student has probably noticed the similarity between an equation and a **formula.** All formulas are equations, and the same rules apply to both. An equation is not called a formula, however, unless it is a statement of a general rule.

7. **Algebra** treats of the equation and its use. Since the use of equations involves the use of letters, it will be necessary, before considering equations, to take up addition, subtraction, multiplication, etc. of expressions in which letters are used.

NOTATION

8. The term **quantity** is used to designate any number that is to be subjected to mathematical processes A quantity is strictly a concrete number; as, 6 books, 5 pounds, 10 yards. *Symbols* used to *represent* numbers, and expres sions containing two or more such symbols, as ax, $10bd$, $(c + 12)$, etc., are often called *quantities*, the term being a convenient one to use.

9. The **signs** $+$, $-$, $\times$, $\div$ are the same in algebra as in arithmetic. The sign of multiplication $\times$ is usually omitted, however, multiplication being indicated by simply writing the quantities together. Thus, abc means $a \times b \times c$; $2xy$ means $2 \times x \times y$ Evidently, the sign $\times$ cannot be omitted between two *figures*, as addition instead of multiplication would then be indicated. Thus, 24 means $20 + 4$ instead of 2×4.

10. A **coefficient** is a figure or letter prefixed to a quantity; it shows how many times the latter is to be taken. Thus, in the expression $4a$, 4 is the coefficient of a and indicates that a is to be taken 4 times; that is, $4a$ is equal to $a + a + a + a$. When several quantities are multiplied together, any of them may be regarded as the coefficient of the others. Thus, in $6axy$, 6 is the coefficient of axy; $6a$, of xy; $6ax$, of y; etc. In general, however, when a coefficient is spoken of, the numerical coefficient only is meant, as the 6 above. When no numerical coefficient is written, it is understood to be 1. Thus, cd is the same as $1cd$.

11. The **factors** of a quantity are the quantities that, when multiplied together, will produce it. Thus, 2, 3, and 3 are the factors of 18, since $2 \times 3 \times 3 = 18$; 2, a, and b are the factors of $2ab$, since $2 \times a \times b = 2ab$.

12. An **exponent** is a small figure placed at the right and a little above a quantity; it shows how many times the latter is to be taken *as a factor*. Thus, $4^3 = 4 \times 4 \times 4 = 64$, the exponent 3 showing that the number 4 is to be used 3 times as a factor; likewise, $a^5 = aaaaa$. Any quantity written without an exponent is understood to have the exponent 1; thus, $b^1 = b$.

13. The difference between a coefficient and an exponent should be clearly understood. A coefficient *multiplies* the quantity which it precedes; it shows that the quantity is to be *added to itself*. Thus, $3a = 3 \times a$, or $a + a + a$. An

exponent indicates that a quantity is to be *multiplied by itself*. Thus, $a^3 = a \times a \times a$. A more complete definition of an exponent will be given later.

14. A **power** is the result obtained by taking a quantity two or more times *as a factor*. For example, 16 is the fourth power of 2, because 2 multiplied by itself until it has been taken four times as a factor produces 16; a^3 is the third power of a, because $a \times a \times a = a^3$.

15. A **root** of a quantity is one of its equal factors. Thus, 2 is a root of 4, of 8, and of 16, since $2 \times 2 = 4$, $2 \times 2 \times 2 = 8$, and $2 \times 2 \times 2 \times 2 = 16$, 2 being one of the equal factors in each case. In like manner, a is a root of a^2, a^3, a^5, etc. The symbol that denotes that the second, or square, root is to be extracted is $\sqrt{\ }$; it is called the **radical sign,** and the quantity under the sign is called the **radical.** For other roots, the same symbol is used but with a figure, called the *index* of the root, written above it to indicate the root. Thus, $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, etc. signify the square root, cube root, fourth root, etc of a. The vinculum is generally used in combination with the radical sign to indicate how much of the expression is governed by the sign. Thus, in an expression like $\sqrt{\ }a + b$ it is understood that the square root of a only is wanted. If, however, it were desired to write the square root of the sum of a and b, the expression would be written $\sqrt{a+b}$, the vinculum extending as far as necessary to indicate how much of the expression was governed by the radical sign. Occasionally, the parenthesis is used instead of the vinculum, but seldom in American textbooks; thus, instead of $\sqrt{a+b}$, it would be written $\sqrt{\ }(a+b)$.

16. The use of the parenthesis, bracket, brace, and vinculum was explained in *Arithmetic*. These symbols are called **symbols of aggregation,** meaning that the quantities enclosed within them are aggregated, or collected, into one quantity.

17. The **terms** of an algebraic expression are those parts that are connected by the signs + and −. Thus, x^2, $-2xy$, and y^2 are terms of the expression $x^2 - 2xy + y^2$. When a term contains both figures and letters, the part consisting of letters is called the **literal** part of the term; thus, xy is the literal part of the term $2xy$.

18. Like terms are those that differ only in their numerical coefficients; all others are **unlike terms.** Thus, $2ab^2$ and $5ab^2$ are like terms; $5ab$ and $5ab^2$ are unlike terms, because one contains b and the other b^2.

19. A **monomial** is an expression consisting of only one term; as, $4abc$, $3x^2$, $2ax^3$, etc.

20. A **binomial** is an expression consisting of two terms; as, $a + b$, $2a + 5b$, etc.

21. A **trinomial** is an expression consisting of three terms; as, $a^2 + 2ab + b^2$, $(a + x)^2 - 2(a + x)y + y^2$, etc., the expression $(a + x)$ being treated as one quantity.

22. A **polynomial** is an expression consisting of more than two terms The name is usually applied only to an expression consisting of four or more terms.

23. The polynomial $a + a^2b + 2a^3 - 3a^4b - a^5$ is said to be arranged according to the *increasing* or *ascending powers* of a, because the exponents of a increase from left to right, the exponent of the first a being 1 understood. (Art. **12.**) The polynomial $a^2b^3 + ab^2 + 4a^4b + 1$ is arranged according to the *decreasing* or *descending powers* of b, the exponents of b decreasing in order from left to right.

24. The arrangement of the terms of a polynomial does not affect its value. Thus, $x^2 + 2xy + y^2$ has the same value as $2xy + y^2 + x^2$, just as $2 + 6 + 4$ has the same value as $6 + 4 + 2$.

31. It really does not matter which quantity is taken as positive and which as negative, so long as the two are opposite in character; but it is customary to call something gained positive and something lost negative. Thus, money earned is usually regarded as positive, money owed as negative; distance up, positive, distance down, negative.

32. The signs + and − may be used in two entirely different senses; heretofore, they have been used exclusively as symbols of operation; thus, + placed between two quantities indicates that they are to be added, etc In the distinction between positive and negative quantities, however, the positive quantity is denoted by the sign + and the negative quantity by the sign −. Hence, under different circumstances, these signs may denote addition and subtraction, or they may denote positive and negative quantities. Suppose we write the expression \$500 − \$200 = \$300; this may mean either \$500 − (+\$200) = \$300, or \$500 + (− \$200) = \$300. In the first case, \$200 is *positive* and is *subtracted* from \$500; in the second case, \$200 is *negative* and is *added* to \$500. The result of the operation, \$300 is the same in either case, as will be shown later. For convenience, therefore, it is always assumed that any algebraic expression consisting of two or more terms invariably represents the *addition of those terms.* Thus, an expression like $a^2 - 2ab + b^2$ is always understood to mean $+a^2 + (-2ab) + (+b^2)$. This fact should be kept in mind, as it will be of assistance later.

33. It is usual to consider that quantities *increase* in a positive direction and *decrease* in a negative direction. For example, when the mercury in a thermometer goes up (rises), the temperature increases, but when the mercury falls, the temperature decreases. This distinction is made, however, only in the manner here indicated; it has nothing to do with the actual numerical value of the quantities. But when, for any purpose, the distinction is made, *any positive quantity, no matter how small, is greater than any*

negative quantity; also, of any two negative quantities, the smaller is the greater. This point is illustrated very nicely by reference to Fig 1. From various points on the circle draw perpendiculars to the diameter $A\,B$. Call any perpendicular above $A\,B$ positive and any below negative. If the point selected corresponds with A or B, the length of the perpendicular is zero. Now if we consider that the chief object is to reach the highest point possible above $A\,B$ without going beyond the circle, it is evident that any perpendicular above $A\,B$, as $a\,a'$, $b\,b'$, or $e\,e'$, is greater than any perpendicular below $A\,B$, as $c\,c'$, $d\,d'$, or $f\,f'$; it is also evident that $f\,f'$ is greater than $c\,c'$ or $d\,d'$, since the point f is nearer the highest point of the circle than c or d. Furthermore, the figure shows that zero is greater than any negative quantity.

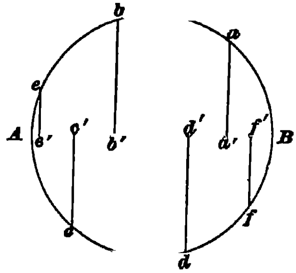


FIG 1

34. When writing algebraic expressions, if a positive term stands alone, or if the first term of an expression is positive, the plus sign is omitted, it being understood that the term is positive. Thus, $3a$ means the same as $+ 3a$, and $a - b$ the same as $+ a - b$. The minus sign must never be omitted.

EXAMPLES FOR PRACTICE

Express the following algebraically:

1. Three x square y square, minus two cd times the quantity a plus b.
Ans. $3x^2y^2 - 2cd(a + b)$

2. The quantity m square plus two mn plus n square in parenthesis, times a square b cube c fourth.
Ans. $(m^2 + 2mn + n^2)a^2b^3c^4$

3. A, plus the square root of D, times the parenthesis X plus Y.
Ans. $A + \sqrt{D}(X + Y)$

4. A, plus the square root of D times the parenthesis X plus Y.
Ans $A + \sqrt{D(X + Y)}$

5 Ten x plus y, minus 7 times the quantity x minus the fraction y over 4 in parenthesis, plus the fraction x square minus y square over two cd.

Ans. $10x+y-7\left(x-\frac{y}{4}\right)+\frac{x^2-y^2}{2cd}$

When $a = 6$, $b = 5$, and $c = 4$, find the numerical values of

6. $a^2 + 2ab + b^2$ Ans. $6^2 + 2 \times 6 \times 5 + 5^2 = 121$
7. $2a^2 + 3bc - 5$. Ans. $72 + 60 - 5 = 127$
8. $2ac^3 - a^2(a + b)$. Ans. 11,892
9. $abc^2 + ab^2c - a^2bc$. Ans. 360

When $x = 8$ and $y = 6$, what do the following equal:

10. $(x+y)(x-y)-\sqrt{\frac{x+y^2}{11}}$? Ans. $(8+6)(8-6)-\sqrt{\frac{8+6^2}{11}} = 26$
11. $\sqrt{(x+y^2)(x^2+y)}-(x-y)(\sqrt[3]{x}+y)$? Ans. 39.5
12. $\frac{x^2y^2}{x+y}+\frac{x^2y(x+y^2)}{\sqrt{3xy}}$? Ans. 1,572 57

ADDITION

ADDITION OF MONOMIALS

35. The operations of addition, subtraction, multiplication, and division performed with algebraic expressions are each based on the same operations performed with monomials; hence, if the latter are clearly understood no trouble will be experienced with the former.

36. There are four cases in connection with addition of two monomials: when both are positive, when the first is positive and the second is negative, when the first is negative and the second is positive, and when both are negative. Let the monomials be the numbers 11 and 6; then the four cases are represented as follows:

$$(+11)+(+6) = 11+6 = +17 \qquad (1)$$
$$(+11)+(-6) = 11-6 = +5 \qquad (2)$$
$$(-11)+(+6) = -11+6 = -5 \qquad (3)$$
$$(-11)+(-6) = -11-6 = -17 \qquad (4)$$

The second form of the above equations follows from the assumption made in Art. **32.**

37. To interpret these results, and, also, those which are obtained from the operations of subtraction, multiplication,

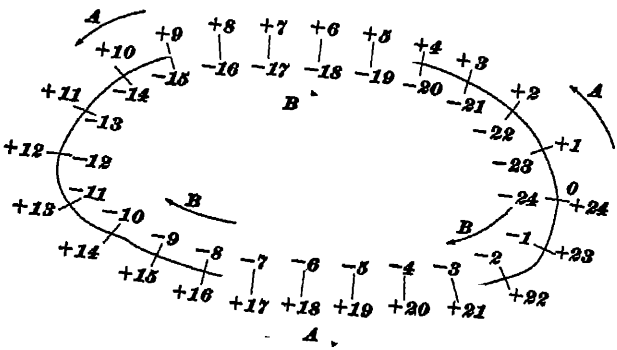


FIG. 2

and division, suppose that a man has a plot of ground, as represented in Fig 2, and that the distance around it is 24 steps of 3 feet each. Suppose, further, that the man is to walk around the plot and that if his face is turned to walk ahead in the direction of the arrows marked A, the action is positive; while if his face is turned to walk ahead in the direction of the arrows marked B, the action is negative. Again, he may walk forwards or backwards; if he walks forwards call the action positive, and if he walks backwards call the action negative.

Having made these assumptions, consider the four cases in order.

Case (1),

$$(+11) + (+6) = +17$$

Referring to Fig. 2, the man walks from O to $+11$; the sign in parenthesis, which is plus, indicates that he is to face to walk in the direction of the arrows marked A, and the sign between the terms, which is plus, indicates that he walked forwards as many steps as are indicated by the second term, that is, 6. As a result he stops at $+17$. Hence, $(+11) + (+6) = 11 + 6 = 17$.

Case (2),

$$(+11)+(-6) = +5$$

He walks from O to $+11$, as before; the minus sign in parenthesis indicates that he is to walk in the direction of the arrows marked B, and the plus sign between the terms indicates that he is to walk forwards 6 steps. He therefore stops at $+5$. Hence, $(+11)+(-6) = 11-6 = +5$.

Case (3),

$$(-11)+(+6) = -5$$

He walks from O to -11; the plus sign in parenthesis indicates that he is to walk in the direction of the arrows marked A, and the plus sign between the terms indicates that he is to walk forwards 6 steps. He therefore stops at -5 Hence, $(-11)+(+6) = -11+6 = -5$.

Case (4),

$$(-11)+(-6) = -17$$

He walks from O to -11; the minus sign in parenthesis indicates that he is to walk in the direction of the arrows marked B, and the plus sign between the terms indicates that he is to walk forwards 6 steps. He therefore stops at -17. Hence, $(-11)+(-6) = -11-6 = -17$.

38. An inspection of the results obtained in four cases just given shows that when the two numbers have like signs, the sum is found by adding, as in arithmetic, and prefixing the common sign; and that when the two numbers have unlike signs, the sum is found by subtracting, as in arithmetic, and prefixing to the result the sign of the greater number.

39. To add two like quantities, as $11a$ and $6a$, whatever the signs may be, simply add the numerical coefficients as above directed and prefix the result to the letters forming the monomial. For example, the sum of $-11a$ and $6a$ is $-5a$, of $-11a$ and $-6a$ is $-17a$, etc.; also, of $-11ab$

and $6ab$ is $-5ab$, of $11ab$ and $6ab$ is $17ab$, etc. That this is so will be readily seen by referring again to Fig. 2. The distance around the plot is 24 steps, and each step is 3 feet long (see Art. **37**); the distance in feet for 11 steps is 33 feet and for 6 steps 18 feet Instead of writing 33 feet and 18 feet, the number of feet in a step may be represented by a, in which case 33 feet becomes $11a$ and 18 feet, $6a$. If, therefore, $11a$ and $6a$ are added, the result must be equal to the sum of 33 feet and 18 feet. Now, as stated above, $11a + 6a = 17a$; substituting for a its value, 3 feet, $17a = 17 \times 3$ feet $= 51$ feet. Also, 33 feet $+ 18$ feet $= 51$ feet A little reasoning will show that the law holds good whatever the signs of the two quantities.

Again, in 1 foot there are 12 inches; hence 33 feet $= 396$ inches, and 18 feet $= 216$ inches. Letting a represent the number of feet in a step and b the number of inches in a foot, $a \times b$, or ab, represents the number of inches in a step Therefore, $11ab + 6ab = 17ab = 17 \times 3 \times 12$ inches $= 612$ inches $= 396$ inches $+ 216$ inches.

40. Only quantities having the same letters affected with the same exponents can be added, i. e., combined into a single term. For example, 7 and -9, $-ab$ and $8ab$, $4a^2b$ and $-2a^2b$, etc. can be added; but 7^2 and -9, $-a^2b$ and $8ab$, $4a^2b$ and $-2ab^2$, etc. cannot be added. The only way in which 7^2 and -9 can be added, is to change the form of 7^2 so as to get rid of the exponent; this is done by raising 7 to the power indicated, obtaining 49, a number having the exponent 1, the same as the exponent of -9. When this is done, $7^2 + (-9) = 7^2 - 9 = 49 - 9 = 40$. When the unlike terms contain letters, however, it is very seldom possible to change the form so as to make them alike, and it then becomes necessary to indicate the addition. Thus, the sum of $-a^2b$ and $8ab$ is written either $-a^2b + 8ab$ or $8ab - a^2b$, according to which arrangement of terms is desired; so also, the sum of $4a^2b$ and $-2ab^2$ is written either $4a^2b - 2ab^2$ or $-2ab^2 + 4a^2b$. But when no

particular arrangement of terms is desired, a positive term is always written first.

Suppose it were required to add the following monomials, a^2b, $-4a^2b$, $-2a^2b$, and $3a^2b$. The sum of the first two is $a^2b - 4a^2b = -3a^2b$, since when no numerical coefficient is written, it is always understood to be 1 Adding to this result the third monomial, the sum is $-3a^2b - 2a^2b = -5a^2b$. Adding to this result the fourth monomial, the sum is $-5a^2b + 3a^2b = -2a^2b$, the sum of all the monomials. The addition may be performed more rapidly and conveniently by adding all the positive and all the negative monomials separately, and then adding the two sums. Thus, $a^2b + 3a^2b = 4a^2b$; $-4a^2b - 2a^2b = -6a^2b$; $-6a^2b + 4a^2b = -2a^2b$.

41. From these illustrations, the following important principle is derived. *If all the terms to be added are positive, the sum is positive; if all are negative, the sum is negative If one term is positive and the other is negative, the sum has the sign of the numerically greater.* If there are several terms to be added, part of which are positive and part negative, the sum is positive or negative according as the sum of the positive terms is numerically greater or less than the sum of the negative terms.

To add like quantities having the same sign:

Rule I.—*Add the coefficients, give the sum the common sign, and annex the common literal part.*

To add like quantities having different signs:

Rule II.—*Add the positive and the negative coefficients separately, and from the greater sum subtract the lesser. Give the remainder the sign of the greater sum, and annex the common literal part.*

EXAMPLE 1 —Find the sum of $-2abxy$, $-abxy$, $-3abxy$, and $-6abxy$.

SOLUTION.—The sum of the coefficients is 12 (remember that the coefficient of $-abxy$ is 1), and the common sign is $-$. The common literal part, $abxy$, annexed to these gives as the result $-12abxy$ (Rule I.)

EXAMPLE 2.—Add xy^2, $-2xy^2$, $8xy^2$, and $-4xy^2$.

SOLUTION —The sum of the coefficients of the positive terms is 9, and of the negative terms, 6 Their difference is 3, and the sign of the greater sum is +. The common literal part, xy^2, annexed to these gives as the result $3xy^2$. (Rule II.)

EXAMPLES FOR PRACTICE

Find the sum of the following:

1. $-6a^2$, $2a^2$, $-5a^2$, $4a^2$, $-3a^2$, and a^2. Ans. $-7a^2$
2. $2a^2b$, $-a^2b$, $11a^2b$, $-5a^2b$, $4a^2b$, and $-9a^2b$. Ans. $2a^2b$
3. $2x^2$, $3xy$, $-x^2$, $8y^2$, $-5xy$, and $-7y^2$. Ans $x^2-2xy+y^2$

NOTE —Combine like terms and connect with respective signs.

4. a^2bc, $-2ab^2c$, $3abc^2$, $-4a^2bc$, and $5ab^2c$

Ans $3ab^2c-3a^2bc+3abc^2$

ADDITION OF POLYNOMIALS

42. Addition of polynomials is merely an extension of addition of monomials.

Rule.—*Write the expressions underneath one another, with like terms in the same vertical column. Add each column separately, and connect the sums by their proper signs.*

EXAMPLE 1 —Find the sum of $5a^2+6ac-3b^2-2xy$, $7ac-3a^2+4b^2+3xy$, and $4xy-5b^2+8ac-a^2$.

SOLUTION.—Writing like terms in the same vertical column, we have

$$\begin{array}{rrrrl} & 5a^2 & +\ 6ac & -3b^2 & -2xy \\ & -3a^2 & +\ 7ac & +4b^2 & +3xy \\ & -\ a^2 & +\ 8ac & -5b^2 & +4xy \\ \textit{sum} & a^2 & +21ac & -4b^2 & +5xy \quad \text{Ans.} \end{array}$$

EXAMPLE 2 —Find the sum of $a^2x-ax^2-x^2$, $ax-x^2-a^2$, $-2a^2-2a^2x-2ax^2$, and $3a^2-3a^2x+3ax^2$.

SOLUTION.—

$$\begin{array}{rrrrrr} & a^2x & -ax^2 & -\ x^2 & & \\ & & & -\ x^2 & -\ a^2 & +ax \\ & -2a^2x & -2ax^2 & & -2a^2 & \\ & -3a^2x & +3ax^2 & & +3a^2 & \\ \textit{sum} & -4a^2x & +0 & -2x^2 & +0 & +ax \end{array}$$

$$= ax-4a^2x-2x^2 \quad \text{Ans. (Arts. 23 and 24.)}$$

EXAMPLES FOR PRACTICE

Find the sum of the following.

1. $ax + 2bx + 4by - 3ay$, $2ax + bx + 2ay - by$, and $4ax + 3by$.
   Ans. $7ax + 3bx + 6by - ay$
2. $a - x + 4y - 3z + w$, $z + 3a - 2x - y - w$, and $x + y + z$.
   Ans $4a - 2x + 4y - z$
3. $2a - 3b + 4d$, $2b - 3d + 4c$, $2d - 3c + 4a$, and $2c - 3a + 4b$.
   Ans $3a + 3b + 3c + 3d$
4. $6x - 3y + 7m$, $2n - x + y$, $2y - 4x - 5m$, and $m + n - y$.
   Ans. $x - y + 3m + 3n$

SUBTRACTION

SUBTRACTION OF MONOMIALS

43. As in addition, there are four cases, as follows, using the same numbers as in Art. **36.**

$$(+11) - (+6) = 11 - 6 = +5 \qquad (1)$$
$$(+11) - (-6) = 11 + 6 = +17 \qquad (2)$$
$$(-11) - (+6) = -11 - 6 = -17 \qquad (3)$$
$$(-11) - (-6) = -11 + 6 = -5 \qquad (4)$$

To interpret these results, refer again to Fig. 2 As before, the sign of the first term, or minuend, indicates whether the man walks first in the direction of the arrows marked A or those marked B, and the number 11 indicates where he stops; the sign of the second term, or subtrahend, indicates whether he is *then* to walk in the direction of the arrows A or the arrows B; and the sign between the terms indicates whether the walk is to be forwards or backwards.

Case (1),
$$(+11) - (+6) = +5$$

He walks from O to $+11$; the sign of the second term being plus, indicates that he is to face so as to walk in the direction of the arrows marked A; the sign between the terms indicates that he is to walk backwards 6 steps. He therefore stops at $+5$ Hence, $(+11) - (+6) = 5$.

Had the subtrahend been numerically greater than the minuend, that is, had the operation been $(+6) - (+11)$, he would have walked first to $+6$ and then backwards 11 steps to -5. Hence, $(+6) - (+11) = -5$.

Case (2),

$$+11 - (-6) = +17$$

He walks first to $+11$; turns around to face as though to walk in the direction of the arrows marked B; then walks backwards 6 steps, stopping at $+17$. Hence, $(+11) - (-6) = +17$. Also, $(+6) - (-11) = +17$.

Case (3),

$$(-11) - (+6) = -17$$

He walks first to -11, turns around to face as though to walk in the direction of the arrows marked A (indicated by the plus sign of the second term); then walks backwards 6 steps, stopping at -17. Hence, $(-11) - (+6) = -17$. Also, $(-6) - (+11) = -17$.

Case (4),

$$(-11) - (-6) = -5$$

He walks to -11; the minus sign of second term indicates he is to face so as to walk in the direction of the arrows marked B; the minus sign between the terms indicates that he is to walk backwards 6 steps. He therefore stops at -5. Hence, $(-11) - (-6) = -5$.

Had the subtrahend been greater numerically than the minuend, that is, had the operation been $(-6) - (-11)$, he would have walked first to -6, and then walked backwards 11 steps, stopping at $+5$ Hence, $(-6) - (-11) = +5$.

44. An inspection of these results shows: first, that when the signs are alike, as in cases (1) and (4), the difference is equal to the *difference* between the two numbers, and the sign of the difference is the same as the sign of the larger number; second, that when the signs are unlike, as in cases (2) and (3), the difference is equal to the *sum* of the two numbers, and the sign of the difference is the same as the sign of the minuend.

45. Consider further the four cases of subtraction. The result obtained in each case might also be obtained by changing the sign of the subtrahend and proceeding as in addition, as follows:

$$\begin{aligned}
(+11) - (+6) &= (+11) + (-6) = 11 - 6 = +5 \\
(+11) - (-6) &= (+11) + (+6) = 11 + 6 = +17 \\
(-11) - (+6) &= (-11) + (-6) = -11 - 6 = -17 \\
(-11) - (-6) &= (-11) + (+6) = -11 + 6 = -5
\end{aligned}$$

Moreover, this method will always produce the same results as will be obtained by applying the law for subtraction stated in Art. **44.** Hence, the following rule for the subtraction of like monomials:

Rule.—*Change the sign of the subtrahend, and proceed as in addition.*

EXAMPLE.—From $-3ab^2x$ take $7ab^2x$.

SOLUTION—Changing the sign of the subtrahend, $7ab^2x$, and *adding*, we have

$$\begin{array}{rl}
-3ab^2x & -\ 3ab^2x \\
7ab^2x & -\ 7ab^2x \text{ (sign changed)} \\
 & -10ab^2x \quad \text{Ans}
\end{array}$$

If the monomials are unlike, the difference cannot be expressed as a single term. Thus, to subtract $5ab^2x$ from $3a^2bx$, change sign of $5ab^2x$ and add it to $3a^2bx$. obtaining $3a^2bx + (-5ab^2x) = 3a^2bx - 5ab^2x$.

46. In arithmetic, subtraction consists in finding how much greater (or less) some number is than another. In algebra, subtraction has an entirely different meaning on account of the use of negative quantities. In algebra, subtraction consists in finding what quantity must be added to the subtrahend to produce the minuend, and the subtrahend may be greater than or less than the minuend, and either may be positive or negative. The result of the subtraction must in all cases be the actual difference, and its sign must show whether the subtrahend must be increased or decreased to produce the minuend. For example, if A has \$11 and B has \$6, + \$5 must be *added* to B's money to

make the amount the same as A's; also, − \$5 must be added to A's money to make the amount the same as B's. In one case the subtrahend is increased, and in the other it is decreased, and the sign of difference shows which occurred. Hence, $11 - 6 = 5$, and $6 - 11 = -5$.

Further, if A has \$11 and B owes \$6, i. e., has − \$6, it is necessary to increase the amount B has + \$17 to make the amount equal to A's, since it would take \$6 to pay what B owes and \$11 more to reach the amount A has. Since the amount B had was increased, the sign of the difference is +. Hence, $11 - (-6) = +17$.

If A owes \$11 and B has \$6, B must lose \$17 in order to owe the same amount as A. Hence, $-11 - 6 = -17$.

If A owes \$11 and B owes \$6, B must lose \$5 more to owe as much as A, and A must gain \$5 to owe as little as B. Hence, $-11 - (-6) = -5$, and $-6 - (-11) = +5$.

EXAMPLES FOR PRACTICE

Solve the following:

1. From $17a$ take $-11a$. Ans. $28a$
2. From $-11a$ take $17a$. Ans. $-28a$
3. Subtract $5cd$ from $-4cd$ Ans. $-9cd$
4. Subtract $-10b^2$ from $-10b^2$. Ans. 0
5. What quantity added to $10xy$ will produce $-12xy$? Ans. $-22xy$
6. What does $10xy$ subtracted from $-12xy$ equal? Ans. $-22xy$

SUBTRACTION OF POLYNOMIALS

47. To subtract one polynomial from another:

Rule.—*Write the subtrahend underneath the minuend, with like terms in the same vertical column. Change the sign of each term of the subtrahend, or imagine the sign of each term to be changed, and proceed as in addition.*

EXAMPLE 1.—From $8ac - 2b$ subtract $ac - b - d$.

SOLUTION.—

$$\begin{array}{lrl} & 3ac - 2b & \\ & -ac + b + d & \text{(subtrahend with signs changed)} \\ \textit{difference} & 2ac - b + d & \text{Ans.} \end{array}$$

EXAMPLE 2 —From $2x^3 - 3x^2y + 2xy^2$ subtract $x^3 - xy^2 + y^3$.

SOLUTION.—

$$\begin{array}{lrrrl} & 2x^3 & -\, 3x^2y & +\, 2xy^2 & \\ & -\; x^3 & & +\; xy^2 & -\, y^3 \text{ (subtrahend with signs changed)} \\ \textit{difference} & x^3 & -\, 3x^2y & +\, 3xy^2 & -\, y^3. \text{ Ans.} \end{array}$$

EXAMPLES FOR PRACTICE

Solve the following:

1. From $7a + 5b - 3c$ take $a - 7b + 5c - 4$. Ans. $6a + 12b - 8c + 4$
2. From $3m - 5n + r - 2s$ take $2r + 3n - m - 5s$.
   Ans $4m - 8n - r + 3s$
3. Subtract $2x - 2y + 2$ from $y - x$. Ans. $3y - 3x - 2$
4. Subtract $3x^3 + 4x^2y - 7xy^2 + y^3 - xy^3$ from $5x^3 + x^2y - 6xy^2 + y^3$.
   Ans. $2x^3 - 3x^2y + xy^2 + xy^3$

SYMBOLS OF AGGREGATION

48. Parentheses, brackets, etc. being used to enclose expressions that are to be treated as one quantity, the sign before the symbol affects the *entire expression*, not the first term only. Thus, $-(a^2 - 2ab + b^2)$ signifies that all the terms are to be subtracted from what precedes, not a^2 only.

49. When combining the terms of any expression without parentheses, proceed as in addition of monomials. When a parenthesis is preceded by a minus sign, the expression within the parenthesis must be considered as a subtrahend, and all signs must be changed before removing the parenthesis.

50. If, on the contrary, the sign before the parenthesis is plus, the signs of the terms within the parenthesis must not be changed when the parenthesis is removed, because the signs of the terms are not changed in addition.

which, in turn, is equal to

$$6a - b + 7cd - 4a + 2cd - a + b$$

Combining like terms

$$6a - 4a - a - b + b + 7cd + 2cd = a + 9cd \quad \text{Ans.}$$

53. If it is desired to enclose several terms in parenthesis or some other symbol of aggregation, and the sign of the first term to be so enclosed is plus, simply write the symbol so as to enclose the desired terms. But if the sign of the first term is minus, it is customary to change the signs of all the terms enclosed and write the minus sign before the parenthesis, so that the first term within the parenthesis may be positive. Thus, if it were desired to enclose the last two terms of $x^2 - 2ax + a^2$ in parenthesis, it would be written $x^2 - (2ax - a^2)$; while if the first two terms were to be enclosed, it would be written $(x^2 - 2ax) + a^2$.

EXAMPLES FOR PRACTICE

Remove the parentheses from the following .

1. $-(2mn - m^2 - n^2)$. Ans $m^2 - 2mn + n^2$
2. $1 - (-b + c + 3)$. Ans. $b - c - 2$
3. $5a - 4b + 3c - (-3a + 2b - c)$ Ans $8a - 6b + 4c$
4. $3x - (2x - 5) + (7 - x)$. Ans. 12

Remove the symbols of aggregation from the following :

5. $m - [4n - k - (m + n - 2k)]$. Ans. $2m - 3n - k$
6. $5x - (2x - 3y) - (x + 5y)$. Ans. $2x - 2y$
7. $3a - [7a - (5a - b - a)] - (-a - 4b)$. Ans $a + 3b$
8. $3x + \{2y - [5x - (3y + \overline{x - 4y})]\}$. Ans. $y - x$
9. $100x - \{200x - [500x - (-100x) - 300x] - 400x\}$. Ans. $600x$
10. $7cx - \{4cy - [(4cx + 3cy) + cy - cx]\}$. Ans $10cx$

NOTE.—Observe that the sign before the parenthesis is + understood.

11. Enclose within parenthesis the second, third, and fourth terms of $x^4 - 4x^3 + 6x^2 - 4x + 1$. Ans. $x^4 - (4x^3 - 6x^2 + 4x) + 1$

12. Enclose the last two terms of $x^4 - 4x^3 + 6x^2 - 4x + 1$ in parenthesis, the last three terms in brackets, and the last four terms in braces. Ans. $x^4 - \{4x^3 - [6x^2 - (4x - 1)]\}$

MULTIPLICATION

MULTIPLICATION OF MONOMIALS

54. Multiplication of algebraic quantities consists of two distinct operations; first, the multiplication of the coefficients, and, second, the multiplication of the literal parts. The second operation will be treated first.

Consider two quantities as a and b; their product is evidently $a \times b = ab$, the sign of multiplication being understood. The product of ab and c is evidently $ab \times c = a \times b \times c = abc$. The product of a and a is $a \times a = a^2$; of a^2 and a, is $a^2 \times a = a \times a \times a = a^3$; of a^2 and b is $a^2 \times b = a^2b$; of a^2b and bc is $a^2b \times bc = a \times a \times b \times b \times c = a^2b^2c$· and of a^3b^2 and a^2b^3c is $a \times a \times a \times b \times b \times a \times a \times b \times b \times b \times c = a \times a \times a \times a \times a \times b \times b \times b \times b \times b \times c = a^5b^5c$. An inspection of these results shows that the product consists of all the letters occurring in both multiplicand and multiplier and that the exponents of the letters in the product are equal to the *sum* of the exponents of the corresponding letters in the multiplicand and multiplier. The law is perfectly general, whether the exponents are positive (as above) or negative, integral (as above) or fractional, provided that the word sum is understood to mean *algebraic sum.*

55. The coefficients are multiplied separately and in the same manner as in arithmetic. For example, $5a^2b^2 \times 3ab^2 = 5 \times a^2b^2 \times 3 \times ab^2 = 5 \times 3 \times a^2b^2 \times ab^2 = 15a^3b^4$; $4ab \times bc = 4 \times ab \times 1 \times bc = 4 \times 1 \times ab \times bc = 4ab^2c$, etc.

56. All that now remains is to determine the sign of the product. As in addition and subtraction, there are four cases as follows:

$$(+11) \times (+6) = +66 \qquad (1)$$
$$(+11) \times (-6) = -66 \qquad (2)$$
$$(-11) \times (+6) = -66 \qquad (3)$$
$$(-11) \times (-6) = +66 \qquad (4)$$

Referring to Fig. 3, which is Fig 2 repeated here for convenience, the man starts walking in all cases at the zero point O. Call 11 in all four cases the multiplicand and 6

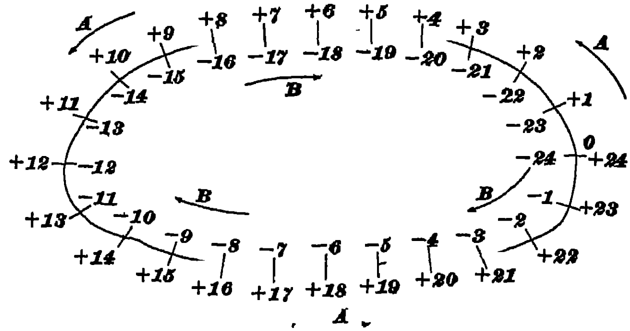


FIG. 3

the multiplier. Let the sign of the multiplicand indicate the direction faced and the sign of the multiplier indicate whether the walk is forwards or backwards.

Case (1),

$$(+11) \times (+6) = +66$$

The man starts at O, faces to walk in the direction of the arrows marked A and takes 66 steps forwards, going twice around the plot and stopping at $+18$. Hence, $(+11) \times (+6) = 11 \times 6 = 66$.

Case (2),

$$(+11) \times (-6) = -66$$

He starts at O, faces to walk in the direction of the arrows marked A, and takes 66 steps backwards, going twice around the plot and stopping at -18. Hence, $(+11) \times (-6) = 11 \times -6 = -66$.

Case (3),

$$(-11) \times (+6) = -66$$

He starts at O, faces to walk in the direction of the arrows marked B, takes 66 steps forwards, going twice

around the plot and stopping at -18. Hence, $(-11) \times (+6) = -11 \times 6 = -66$.

Case (4),

$$(-11) \times (-6) = +66$$

He starts at O, faces to walk in the direction of the arrows marked B, takes 66 steps backwards, going twice around the plot, and stopping at $+18$. Hence, $(-11) \times (-6) = -11 \times -6 = +66$.

57. An inspection of the results obtained shows that when the signs of the multiplicand and multiplier are alike, the product is positive, and when the signs are unlike, the product is negative.

58. A little consideration will show, further, that when a series of monomials are to be multiplied together to form a single term, the sign of the product will depend on the number of minus signs, being positive when the number of negative monomials is even and negative when the number of negative monomials is odd. For example, the product $a \times -b \times c \times -a^2$ is positive, since the number of minus signs is *two*, an even number. This can be shown by actual multiplication; thus, $a \times -b \times c \times -a^2 = -ab \times c \times -a^2 = -abc \times -a^2 = a^3bc$. Again, the product $a \times -b \times c \times -a^2 \times -b$ is negative, since the number of minus signs is *three*, an odd number. This can also be shown by actual multiplication; thus, $a \times -b \times c \times -a^2 \times -b = -ab \times c \times -a^2 \times -b = -abc \times -a^2 \times -b = a^3bc \times -b = -a^3b^2c$.

59. From the foregoing the following rule is obtained for the multiplication of monomials:

Rule.—*To the product of the coefficients, annex the letters of both monomials; give each letter an exponent equal to the sum of the exponents of that letter.*

Make the sign of the product plus, when the signs of the multiplicand and multiplier are alike; and minus, when they are unlike.

EXAMPLE.—Multiply $4a^2b$ by $-5a^3bc$.

SOLUTION.—The product of the coefficients is 20, and the letters to be annexed are a, b, and c. The new exponent of a is 5, and of b, 2, since $a^{2+3} = a^5$, and $b^{1+1} = b^2$ The sign of the product is minus, since the two factors have different signs

Hence, $4a^2b \times -5a^3bc = -20a^5b^2c$ Ans

60. When there are more than two factors, there are simply three or more examples in multiplication to solve in succession, each to be performed by the foregoing rule. Or, multiply the coefficients as in arithmetic, write all the letters that occur in the factors, and give to each an exponent equal to the sum of the exponents of the letters in the factors. Determine the sign by the principle given in Art. **58.**

EXAMPLE.—Find the continued product of $6x^2yz^3$, $-9x^2y^2z^2$, and $-3x^4yz$.

SOLUTION —*First* · $6x^2yz^3 \times -9x^2y^2z^2 = -54x^{2+2}y^{1+2}z^{3+2}$, or $-54x^4y^3z^5$. Now, multiplying this product by $-3x^4yz$, we have $-54x^4y^3z^5 \times -3x^4yz = 162x^8y^4z^6$. Ans.

Second · The product of the coefficients is $6 \times 9 \times 3 = 162$. The sum of the exponents of x is $2+2+4 = 8$, of y is $1+2+1 = 4$, and of z is $3+2+1 = 6$. Since the number of minus signs is even, the sign of the product is +. Hence, the product is $162x^8y^4z^6$. Ans.

EXAMPLES FOR PRACTICE

Find the product of·

1. a^3b^2 and $-5abd$ Ans. $-5a^4b^3d$
2. $-7xy$ and $-7x^2y^2$. Ans $49x^3y^3$
3. $-15m^5n^6$ and $3mn$. Ans. $-45m^6n^7$
4. $3a(x-y)^2$ and $2a^2(x-y)$. Ans. $6a^3(x-y)^3$

SUGGESTION —Treat the $(x-y)$ as though it were a single letter

5. Find the continued product of $2a^3m^2x$, $-3a^2mx^3$, and $4am^3x^2$. Ans. $-24a^6m^6x^6$
6. What does $-a^2bn \times -2cdn \times -3bdc^2 \times -2acn^2$ equal? Ans. $12a^3b^2c^4d^2n^4$

MULTIPLICATION OF POLYNOMIALS

61. There are two cases, (1) when the multiplier is a monomial, and (2) when it contains more than one term. The first case will be considered first. The process is best illustrated by an example.

EXAMPLE.—Multiply $3a^2b^2 - 3a^4b - b^4 + a^5 - 4ab^3$ by $-5ab^2$.

SOLUTION —*First:*

$$\begin{array}{l} \quad 3a^2b^2 - 3a^4b - b^4 + a^5 - 4ab^3 \\ -\ 5ab^2 \\ \hline -15a^3b^4 + 15a^5b^3 + 5ab^6 - 5a^6b^2 + 20a^2b^5 \end{array}$$

Arranging terms according to descending powers of a, the product is $-5a^6b^2 + 15a^5b^3 - 15a^3b^4 + 20a^2b^5 + 5ab^6$ Ans.

Second. Arranging the multiplicand according to descending powers of a, before multiplying,

$$\begin{array}{l} \quad a^5 - 3a^4b + 3a^2b^2 - 4ab^3 - b^4 \\ -5ab^2 \\ \hline -5a^6b^2 + 15a^5b^3 - 15a^3b^4 + 20a^2b^5 + 5ab^6 \quad \text{Ans.} \end{array}$$

EXPLANATION.—For convenience, the multiplication is begun with the left-hand term of the multiplicand instead of at the right, as in arithmetic, and the multiplier is written at the left also. Each term of the multiplicand is then treated as a monomial and multiplied by the multiplier, according to the rule of Art. **59,** and the various results are added algebraically, as indicated As it is more convenient, and in most cases necessary, to have the resulting product arranged according to the descending or ascending powers of one of the letters, the terms are then rearranged according to the descending powers of a.

A better way to obtain the result indicated by the answer is to arrange the multiplicand according to the descending or ascending powers of the letter selected, by which means the product will not require to be rearranged; this is indicated by the second solution. Here, $a^5 \times -5ab^2 = -5a^6b^2$, $-3a^4b \times -5ab^2 = 15a^5b^3$, $3a^2b^2 \times -5ab^2 = -15a^3b^4$, $-4ab^3 \times -5ab^2 = 20a^2b^5$, and $-b^4 \times -5ab^2 = 5ab^6$.

62. From the foregoing, the following rule is derived:

Rule.—*Arrange the terms of the multiplicand according to the descending or ascending powers of some letter, and multiply each term of the multiplicand by the monomial multiplier; the algebraic sum of the results will be the product sought.*

63. When the multiplier consists of more than one term:

Rule.—*The terms of the multiplicand are arranged according to the descending or ascending powers of one of the letters, and the terms of the multiplier are arranged similarly, with the left-hand term of the multiplier under the left-hand term of the multiplicand. Each term of the multiplicand is then multiplied by the first (left-hand) term of the multiplier, proceeding from left to right, and the successive results are written underneath, connected by their proper signs, for the first partial product. Each term of the multiplicand is then multiplied by the second term of the multiplier for the second partial product, the terms similar to those in the first partial product being placed underneath to form a column. The multiplication is thus continued with the third and remaining terms until all the terms of the multiplier have been used as monomial multipliers. The various columns are then added and the result is the product sought.*

Example 1.—Multiply $6a - 4b$ by $4a - 2b$.

Solution.—Write the multiplier under the multiplicand, and begin to multiply *at the left*

	$6a - 4b$	(1)
	$4a - 2b$	
Multiplying (1) by $4a$ gives	$24a^2 - 16ab$	(2)
Multiplying (1) by $-2b$ gives	$-12ab + 8b^2$	(3)
Adding (2) and (3) gives	$24a^2 - 28ab + 8b^2$	Ans.

The like terms, $-16ab$ and $-12ab$, are written under each other, so that it will be easier to add them

Example 2.—Multiply $x^3 - x + 1 + x^4$ by $1 - x^2 + x$.

Solution.—Arrange both multiplicand and multiplier according to the increasing or the decreasing powers of the same letter, in this case according to the increasing powers of x.

$$
\begin{array}{lrl}
 & 1 - x + x^2 + x^3 & (1) \\
 & 1 + x - x^2 & \\
\text{Multiplying (1) by 1 gives} & 1 - x + x^2 + x^3 & (2) \\
\text{Multiplying (1) by } + x \text{ gives} & x - x^2 + x^3 + x^4 & (3) \\
\text{Multiplying (1) by } - x^2 \text{ gives} & - x^2 + x^3 - x^4 - x^5 & (4) \\
\text{Adding (2), (3), and (4) gives} & 1 \quad - x^2 + 3x^3 \qquad - x^5 & \text{Ans}
\end{array}
$$

EXAMPLE 3.—Find the product of $3a^2b + a^3 + 3ab^2 + b^3$ and $a^3 - b^3 + 3ab^2 - 3a^2b$.

SOLUTION —Arranging the terms according to the descending powers of a and multiplying,

$$
\begin{array}{lllllll}
a^3 + 3a^2b + 3ab^2 + b^3 & & & & & & \\
a^3 - 3a^2b + 3ab^2 - b^3 & & & & & & \\
a^6 & + 3a^5b & + 3a^4b^2 & + a^3b^3 & & & \\
 & - 3a^5b & - 9a^4b^2 & - 9a^3b^3 & - 3a^2b^4 & & \\
 & & 3a^4b^2 & + 9a^3b^3 & + 9a^2b^4 & + 3ab^5 & \\
 & & & - a^3b^3 & - 3a^2b^4 & - 3ab^5 & - b^6 \\
a^6 & + 0 & - 3a^4b^2 & + 0 & + 3a^2b^4 & + 0 & - b^6
\end{array}
$$

Hence, the product is $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$. Ans

64. Multiplication is frequently indicated by enclosing each of the quantities to be multiplied in a parenthesis. The sign of multiplication is not placed between the parentheses, multiplication being understood. When the quantities are multiplied together, the expression is said to be **expanded.**

For example, in the expression $(m - 2n)(2m - n)$, the binomial $m - 2n$ is to be multiplied by the binomial $2m - n$ Performing the multiplication, the product is $2m^2 - 5mn + 2n^2$, which is the expanded form of the expression.

EXAMPLES FOR PRACTICE

Multiply the following

1. $x^2 + 2xy + y^2$ by $x + y$ Ans $x^3 + 3x^2y + 3xy^2 + y^3$
2. $3ab^2m^3 + 4a^2b - 2$ by $a^6b^7m^8$. Ans $3a^7b^9m^{11} + 4a^8b^8m^8 - 2a^6b^7m^8$
3. $c^2 - d^2$ by $c^2 + d^2$. Ans $c^4 - d^4$
4. $x^4 + x^2y^2 + y^4$ by $x^2 - y^2$. Ans. $x^6 - y^6$
5. $3a^2 - 7a + 4$ by $2a^2 + 9a - 5$. Ans. $6a^4 + 13a^3 - 70a^2 + 71a - 20$

Expand the following.

6. $(2a - 3c)(4 - 3a)$. Ans. $8a - 12c - 6a^2 + 9ac$

7. $(x + 2)(x - 2)(x^2 + 4)$. Ans. $x^4 - 16$

8 $[x(x^2 - y^2) - 2][x(x^2 + y^2) + 2]$.

NOTE — The expressions in the brackets reduce to $x^3 - xy^2 - 2$ and $x^3 + xy^2 + 2$. The product of these is $x^6 - x^2y^4 - 4xy^2 - 4$. Ans.

DIVISION

DIVISION OF MONOMIALS

65. The law of signs for division is the same as for multiplication; i. e., *when the dividend and divisor have like signs, the sign of the quotient is plus, and when they have unlike signs, the sign of the quotient is minus.* This may be proved (1) directly or (2) as following from the law of signs for multiplication. It will first be proved directly.

66. There are four cases as follows:

$$(+66) \div (+11) = +6 \qquad (1)$$
$$(+66) \div (-11) = -6 \qquad (2)$$
$$(-66) \div (+11) = -6 \qquad (3)$$
$$(-66) \div (-11) = +6 \qquad (4)$$

Referring to Fig. 3, suppose our man to start from O in all cases. He is to walk 11 steps and count 1, walk 11 steps more and count 2, and so on until he has walked 66 steps. The number of 11-step periods counted will be the quotient. If he walks around the plot in the direction of the arrows A, the quotient will be plus, while if the walk is in the direction of the arrows B, the quotient will be minus. Let the sign of the dividend indicate the direction he is to face, and let the sign of the divisor indicate whether he is to walk forwards or backwards.

Case (1),

$$(+66) \div (+11) = +6$$

The plus sign of the dividend shows that he is to face to walk in the direction of the arrows marked A; the plus sign

of the divisor shows he is to walk forwards; hence, he walks around the plot in the direction of the arrows A, and the sign of the quotient is plus. Therefore, $66 \div 11 = 6$.

Case (2),

$$(+66) \div (-11) = -6$$

He faces to walk in the direction of the arrows marked A; the minus sign of the divisor indicates he is to walk backwards; hence, he walks around the plot in the direction of the arrows B, and the sign of the quotient is minus. Therefore, $66 \div -11 = -6$.

Case (3),

$$(-66) \div (+11) = -6$$

The minus sign of the dividend shows he is to face to walk in the direction of the arrows marked B; the plus sign of the divisor indicates he is to walk forwards; hence, he walks around the plot in the direction of the arrows B, and the sign of the quotient is minus. Therefore, $-66 \div 11 = -6$.

Case (4),

$$(-66) \div (-11) = +6$$

He faces to walk in the direction of the arrows B; the minus sign of the divisor indicates he is to walk backwards; hence, he walks around the plot in the direction of the arrows A, and the sign of the quotient is plus. Therefore, $-66 \div -11 = 6$.

67. The second proof follows directly from the laws of multiplication and the fact that the product of the divisor and quotient plus the remainder, if any, must equal the dividend.

Case (1),

$$(+66) \div (+11) = +6$$

Here the product of the divisor 11 and the quotient 6 must equal $+66$. Since only the product of like signs is positive and the sign of the divisor is plus, the sign of the quotient must also be plus.

73. From the foregoing, the following rule is derived for division of monomials:

Rule.—*Divide the coefficient of the dividend by the coefficient of the divisor and to the quotient annex the letters of the dividend, each with an exponent equal to its exponent in the dividend minus its exponent in the divisor, omitting those letters whose exponents become zero.*

Make the sign of the quotient plus when the dividend and divisor have like signs, and minus when they have unlike signs.

EXAMPLE 1.—Divide $6a^5b^4c^3$ by $-3a^2bc^3$.

SOLUTION.—The quotient of $6 \div 3$ is 2. The letters to be annexed, and their exponents, are $a^{5-2} = a^3$, and $b^{4-1} = b^3$. The c has an exponent of $3 - 3 = 0$, so that it becomes equal to 1, and is omitted. The sign of the quotient is minus

Hence, $6a^5b^4c^3 \div -3a^2bc^3 = -2a^3b^3$ Ans.

PROOF — $-3a^2bc^3 \times -2a^3b^3 = 6a^5b^4c^3$

EXAMPLE 2.—Divide $-10a^6b^3c^2d$ by $-2ab^3c$.

SOLUTION.— $-10a^6b^3c^2d \div -2ab^3c = 5a^{6-1}b^{3-3}c^{2-1}d = 5a^5cd$. Ans.

EXAMPLES FOR PRACTICE

Divide the following:

1. $12m^2n$ by $4n$ — Ans. $3m^2$
2. $30x^6y^5bc^3$ by $-6x^5y^5c^2$. — Ans $-5xbc$
3. $-44a^3b^3c^3$ by $-11ab^2c^3$. — Ans $4a^2b$
4. $-100x^4y^3z^2$ by x^3y^2. — Ans. $-100xyz^2$
5. $75pq^2x^3m^4$ by $75x^3$. — Ans. pq^2m^4

DIVISION OF POLYNOMIALS

74. When the divisor is a monomial:

Rule.—*Divide each term of the dividend by the divisor, and connect the partial quotients by their proper signs.*

EXAMPLE.—Divide $12a^2b^4 - 9ab^3 + 6a^3b^4$ by $3ab^3$.

SOLUTION.— $3ab^3)12a^2b^4 - 9ab^3 + 6a^3b^4$

quotient $4ab \quad -3 \quad +2a^2b$ Ans.

EXAMPLES FOR PRACTICE

Divide the following.

1. $64m^2n^3 - 32mn^2 + 8m^2n$ by $8mn$ Ans. $8mn^2 - 4n + m$
2. $27x^3y^2z - 9x^3yz^2 - 333x^3y^2z^2$ by $-3x^3yz$.
   Ans. $-9y + 3z + 111yz$
3. $10(x+y)^2 - 5a(x+y) + 5a^2(x+y)$ by $5(x+y)$.
   Ans $2(x+y) - a + a^2$

75. The division of a polynomial by a polynomial is performed in the same manner as is the operation called *long division* in arithmetic. The work is performed to the best advantage if the dividend and divisor are arranged according to the ascending or descending powers of the same letter. The process is shown in the following example.

EXAMPLE.—Divide $x^3 - 9x^2 + x^4 - 16x - 4$ by $4 + x^2 + 4x$.

SOLUTION.—Arrange the dividend and divisor according to descending powers of x

dividend	$x^4 + x^3 - 9x^2 - 16x - 4$	$(x^2 + 4x + 4$ *divisor*
	$x^4 + 4x^3 + 4x^2$	$(x^2 - 3x - 1$ *quotient*
first new dividend	$-3x^3 - 13x^2 - 16x$	
	$-3x^3 - 12x^2 - 12x$	
second new dividend	$-x^2 - 4x - 4$	
	$-x^2 - 4x - 4$	

Divide the first term of the dividend x^4 by the first term of the divisor x^2 for the first term x^2 of the quotient. Multiply the whole divisor by x^2 and the product is $x^4 + 4x^3 + 4x^2$. Subtract this from the dividend and the remainder is the first new dividend $-3x^3 - 13x^2 - 16x - 4$. The term -4 need not be brought down, since the divisor consists of three terms only.

Divide the first term of the remainder $-3x^3$ by the first term of the divisor x^2 and the result is $-3x$, the second term of the quotient. Again, multiply the whole divisor by this term of the quotient and subtract the product, $-3x^3 - 12x^2 - 12x$, from the first remainder. The remainder is $-x^2 - 4x - 4$, the term -4 being brought down from the original dividend. Divide the first term of this remainder $-x^2$ by the first term of the divisor x^2 and the quotient -1 is the third term of the quotient. Multiply the whole divisor by this term of the quotient and the product is $-x^2 - 4x - 4$. When this product is subtracted from the remainder, $-x^2 - 4x - 4$, there is no remainder.

The sum of the various products plus the remainder, if any, $x^4 + 4x^3 + 4x^2$, $-3x^3 - 12x^2 - 12x$, and $-x^2 - 4x - 4$, is the original dividend.

76. The student will find it advantageous to place the divisor on the right of the dividend, with the quotient below, as shown in the last example It will then be easier to multiply each term of the divisor by the new term of the quotient and there will be less liability of mistakes. The solution will also require less space.

77. To divide one polynomial by another:

Rule.—I. *Arrange both dividend and divisor according to ascending or descending powers of some common letter.*

II. *Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.*

III. *Multiply the entire divisor by the first term of the quotient; write the product under the dividend and subtract it from the dividend for the first remainder.*

IV. *Regard the remainder as a new dividend and divide its first term by the first term of the divisor for the second term of the quotient*

V. *Multiply the whole divisor by the second term of the quotient; write the product under the first remainder and subtract as before.*

VI. *So continue until the remainder is 0, or until the first term of the remainder cannot be divided by the first term of the divisor without a change of sign in the exponent of the quotient.*

NOTE —When there is a final remainder, it is to be written over the divisor and annexed to the quotient.

EXAMPLE 1 —Divide $x^4 + 57x - 70$ by $x^2 + 3x - 5$.

SOLUTION —

dividend	$x^4 \qquad\qquad + 57x - 70($	$x^2 + 3x - 5$ *divisor*
$(x^2 + 3x - 5) \times x^2$	$= x^4 + 3x^3 - 5x^2$	$(x^2 - 3x + 14$ *quotient*
first new dividend	$-3x^3 + 5x^2 + 57x$	
$(x^2 + 3x - 5) \times (-3x) =$	$-3x^3 - 9x^2 + 15x$	
second new dividend	$+ 14x^2 + 42x - 70$	
$(x^2 + 3x - 5) \times 14 \quad =$	$+ 14x^2 + 42x - 70$	

CHECK.—By the definition of division the dividend is the product of the divisor and quotient; therefore, to check a division, multiply the divisor by the quotient, and if the product is the dividend, the work is probably correct. Thus, example 1 may be checked as follows:

$$
\begin{array}{lllll}
\textit{divisor} & x^2 + 3x - 5 & & & \\
\textit{quotient} & x^2 - 3x + 14 & & & \\
 & x^4 + 3x^3 - 5x^2 & & & \\
 & \quad - 3x^3 - 9x^2 + 15x & & & \\
 & \qquad\quad + 14x^2 + 42x - 70 & & & \\
\textit{product} & x^4 \qquad\qquad\quad + 57x - 70 & \textit{dividend} & &
\end{array}
$$

EXAMPLE 2.—Divide (*a*) $x^2 - y^2$ by $x - y$; also, (*b*) $a^4 - x^4$ by $a + x$.

SOLUTION.—

(*a*)

$$
\begin{array}{l}
x^2 - y^2 \,(x - y \\
x^2 - xy \quad (x + y \\
\quad + xy - y^2 \\
\quad + xy - y^2
\end{array}
$$

(*b*)

$$
\begin{array}{l}
a^4 - x^4 \,(a + x \\
a^4 + a^3x \quad (a^3 - a^2x + ax^2 - x^3 \\
\quad - a^3x - x^4 \\
\quad - a^3x - a^2x^2 \\
\qquad\qquad a^2x^2 - x^4 \\
\qquad\qquad a^2x^2 + ax^3 \\
\qquad\qquad\quad - ax^3 - x^4 \\
\qquad\qquad\quad - ax^3 - x^4
\end{array}
$$

Ans. $\begin{cases} (a) & x + y \\ (b) & a^3 - a^2x + ax^2 - x^3 \end{cases}$

EXAMPLE 3.—Divide $a^5 - 9 + 7a^2 - 17a^3$ by $3 + a^2 + 5a$.

SOLUTION.—

$$
\begin{array}{l}
a^5 \qquad\quad - 17a^3 + 7a^2 - 9 \,(a^2 + 5a + 3 \\
a^5 + 5a^4 + 3a^3 \qquad\qquad (a^3 - 5a^2 + 5a - 3 \quad \text{Ans.} \\
\quad - 5a^4 - 20a^3 + 7a^2 \\
\quad - 5a^4 - 25a^3 - 15a^2 \\
\qquad\qquad 5a^3 + 22a^2 - 9 \\
\qquad\qquad 5a^3 + 25a^2 + 15a \\
\qquad\qquad\quad - 3a^2 - 15a - 9 \\
\qquad\qquad\quad - 3a^2 - 15a - 9
\end{array}
$$

EXAMPLES FOR PRACTICE

Divide:

1. $a^2 + 2ab + b^2$ by $a + b$. Ans. $a + b$
2. $a^2 - 2ab + b^2$ by $a - b$. Ans. $a - b$
3. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$. Ans. $a^2 - 2ab + b^2$

4. $a^3 + a^2 + 4a - 20$ by $a - 2$. Ans. $a^2 + 3a + 10$

5. $8x^2 - 26xy + 15y^2$ by $4x - 3y$. Ans. $2x - 5y$

6. $x^4 - x^2y^2 - 12y^4$ by $x^2 + 3y^2$. Ans. $x^2 - 4y^2$

7. $a^4 + 64$ by $a^2 + 4a + 8$. Ans. $a^2 - 4a + 8$

8. $x^2y^2 + 2xy^2z - x^2z^2 + y^2z^2$ by $xy + xz + yz$. Ans. $xy - xz + yz$

9. $12x^4 - 26x^3y - 8x^2y^2 + 10xy^3 - 8y^4$ by $3x^2 - 2xy + y^2$.
Ans. $4x^2 - 6xy - 8y^2$

10. $a^{10} - 29a^4 + 62a^3 + 3a^2 + 16a - 352$ by $a^4 + 2a^3 - 5a - 11$.
Ans. $a^6 - 2a^5 + 4a^4 - 3a^3 + 7a^2 - 16a + 32$

ELEMENTS OF ALGEBRA

(PART 2)

FACTORS AND MULTIPLES

FACTORING

1. It was stated in Art. **4,** Part 1, that factoring is a particularly important operation. The reason that this is so is that terms cannot be combined in algebra as in arithmetic, because the equivalence of the terms is not known. The idea can be best illustrated by an example.

Suppose it is required to multiply 5,402 by 136. The number 5,402 is equal to $5{,}000 + 400 + 2 = 5\,(10^3) + 4\,(10^2) + 2$. Similarly, 136 is equal to $10^2 + 3\,(10) + 6$. If now a be substituted for 10, the two numbers become $5a^3 + 4a^2 + 2$ and $a^2 + 3a + 6$. Multiplying these two algebraic expressions, the product is

$$\begin{array}{rrrrrr}
 & & 5a^3 & + 4a^2 & & + 2 \\
 & & a^2 & + 3a & & + 6 \\
\hline
5a^5 & + 4a^4 & & + 2a^2 & & \\
 & 15a^4 & + 12a^3 & & + 6a & \\
 & & 30a^3 & + 24a^2 & & + 12 \\
\hline
5a^5 & + 19a^4 & + 42a^3 & + 26a^2 & + 6a & + 12
\end{array}$$

That this result is correct can be seen at once by substituting 10 for a, thus:

EXAMPLE 2.—Find the prime factors of 862

SOLUTION.—After dividing 862 by 2, try to divide 431 by 2, 3, 5, 7, 11, 13, 17, and 19. It is unnecessary to try any prime number greater than 19, for 431 divided by 19 gives a quotient less than 23, the next prime number. Therefore, if 431 were divisible by 23 or any number greater than 23, the quotient would be less than 19, and 431 would have a factor less than 19. But by trial it is found that 431 has no factor less than 19, and is, therefore, a prime number. Thus,

$$\begin{array}{r} 2\underline{)862} \\ 431 \end{array}$$

$$862 = 2 \times 431 \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Find the prime factors of:

1. 35. Ans. 5×7
2. 117. Ans. $3^2 \times 13$
3. 3,575. Ans. $5^2 \times 11 \times 13$
4. 13,260. Ans. $2^2 \times 3 \times 5 \times 13 \times 17$

FACTORS OF ALGEBRAIC EXPRESSIONS

11. An algebraic term is **integral** if it does not contain a *letter* as a *divisor*; otherwise it is **fractional.**

Thus, ab, x^2, $3mn^2x$, $\frac{2}{3}xy$ are integral terms; while $a \div b$, $\frac{x}{y}$, $\frac{3m}{4n}$ are fractional. An integral term may have either an integral or a fractional value; so also may a fractional term.

The classification of terms into integral and fractional has reference to their literal part, not to their numerical part or to their numerical value.

An **integral expression** is an expression of which all the terms are integral.

Thus, $5x + 3x^2 + 6x^3 + 3acx^4$ is an integral expression. But $\frac{x}{y} + \frac{3a}{b} + \frac{4m}{x}$ is a fractional expression.

An expression is said to be *integral with respect to a certain letter* when that letter does not occur as a divisor in any term.

Thus, $\frac{x}{a}+\frac{x^2y}{a^2}+\frac{x^3y^2}{a^4}$ is integral with respect to x, but fractional with respect to a. In the following discussion of factors and multiples, only integral expressions are treated.

12. Factors of Monomials.—Since monomials containing more than one element are simply indicated multiplications, the factors of a monomial are found by mere inspection Thus,

$$11a^4x^2 = 11\times a\times a\times a\times a\times x\times x$$

13. Factors of Polynomials.—The product of two or more binomials or trinomials often assumes a certain type form, and when these type forms appear, it is easy to find the factors. Some of the simplest methods of finding these factors are given in the following articles.

CASE I

14. To factor a polynomial when all of its terms have a common factor.—The common factor is found by inspection and the other factor is found by dividing the polynomial by the common factor.

EXAMPLE.—Find the factors of $16x^2y^2 + 4x^3y^2 - 12xy^4$.

SOLUTION —It is evident that each term contains the common factor $4xy^2$. Dividing the number by $4xy^2$, the quotient is $4x + x^2 - 3y^2$, which is the other factor.

Hence, $16x^2y^2 + 4x^3y^2 - 12xy^4 = 4xy^2(4x + x^2 - 3y^2)$ Ans.

15. To discover the monomial factor of a polynomial, first ascertain the factors common to all the numerical coefficients. This is done by ascertaining if the smallest numerical coefficient is contained in the coefficients of *all* the other terms, and if so, reserve it for the coefficient of the monomial factor; if not,

and it is not a prime number, resolve it into its prime factors and see if any are factors of all the remaining terms, multiplying all the common prime factors for the numerical coefficient of the monomial factor sought. Then examine the polynomial to find the letters common to every term, take each of these common letters with the lowest exponent it has in any term of the polynomial. The product of the letters so chosen and the common factors of the numerical coefficients is the monomial factor.

EXAMPLE 1.—Find the factors of $12ab^2c^3 - 18a^3c^2y + 24a^2c^4 - 36a^4bc^5y^2$.

SOLUTION.—The numerical coefficients are 12, 18, 24, and 36. The smallest coefficient, 12, is not a factor of 18; hence, it is resolved into its prime factors, which are $2 \times 2 \times 3$. Since 18 is divisible by one 2 and by 3, and the other coefficients, 24 and 36, are divisible by 12, the numerical coefficient of the monomial factor is $2 \times 3 = 6$, the largest factor common to 12, 18, 24, and 36. The letters a and c are common to all the terms, and the lowest power of a is the first, and of c the square. Therefore, the monomial factor is $2 \times 3 \times a \times c^2$, or $6ac^2$ Dividing the polynomial by $6ac^2$, the quotient is $2b^2c - 3a^2y + 4ac^2 - 6a^3bc^3y^2$. Hence,

$$12ab^2c^3 - 18a^3c^2y + 24a^2c^4 - 36a^4bc^5y^2 = 6ac^2(2b^2c - 3a^2y + 4ac^2 - 6a^3bc^3y^2)$$ Ans.

EXAMPLE 2 —Factor $2ax - bx$

SOLUTION.—The only letter or number common to the two terms is x.

Hence, $2ax - bx = (2a - b)x$ Ans.

EXAMPLES FOR PRACTICE

Factor the following expressions.

1. $a^4 + ax$. Ans. $a(a^3 + x)$
2. $12a^5 - 2a^3 + 4a^4$. Ans. $2a^3(6a^2 - 1 + 2a)$
3. $30m^4n^2 - 6n^3$. Ans. $6n^2(5m^4 - n)$
4. $16x^2y^3 - 8x^5 + 8$. Ans. $8(2x^2y^3 - x^5 + 1)$
5. $4x^3y - 12x^2y^2 + 8xy^3$. Ans $4xy(x^2 - 3xy + 2y^2)$
6. $49a^2b^3c^4 - 63a^3b^2c^4 + 7a^4b^2c^3$. Ans. $7a^2b^2c^3(7bc - 9ac + a^2)$

EQUAL FACTORS

16. Equal factors are those whose terms have the same letters, and whose letters have the same exponents and the same signs. Thus, $5a(2y - x)$ and $5a(2y - x)$ are equal factors of $5a(2y - x) \times 5a(2y - x) = 25a^2(2y - x)^2$; but $5a(2y - x)$ and $-5a(2y - x)$ are unequal factors, since the signs of $5a$ are not the same in both expressions.

17. A product of two equal factors is a **perfect square.** Either of the equal factors of a quantity is called its **square root.**

18. A product of three equal factors is a **perfect cube.** Any one of the three equal factors of a quantity is called its **cube root.**

19. In factoring, it is important to be able to easily distinguish quantities that are perfect squares and cubes, and to determine their roots. By definition, $9a^2b^2$ is a perfect square because $3ab \times 3ab = 9a^2b^2$, and $3ab$ is its square root. Also, $8a^6$ is a perfect cube because $2a^2 \times 2a^2 \times 2a^2 = 8a^6$, and $2a^2$ is its cube root. In each of these cases the coefficients of the roots are multiplied together, and the exponents added, to produce a perfect power. *Hence, a quantity is a perfect square when its coefficient is a perfect square, and the exponents of all its letters can be divided by 2.* For example, $36x^{10}$, $49b^2c^4d^8$, $16a^6b^{12}$, and 1 are all perfect squares, whose roots are $6x^5$, $7bc^2d^4$, $4a^3b^6$, and 1, respectively. No perfect square, however, can have a minus sign; for, let a = any quantity, $-a \times -a = a^2$, and $a \times a = a^2$. The square root of a^2 may be $-a$, or a, and a square root is often written $\pm a$, read plus or minus a.

A quantity is a perfect cube when its coefficient is a perfect cube, and the exponents of all its letters can be divided by 3. Thus, $27x^{12}$, $-64b^3c^9d^6$, $8a^{15}b^{18}$, and 1 are all perfect cubes, whose roots are $3x^4$. $-4bc^3d^2$, $2a^5b^6$, and 1, respectively. The sign of the cube root is always the same as that of its cube.

CASE II

20. To factor a trinomial that is a perfect square.

Any trinomial is a perfect square when the first and the last terms are perfect squares and positive, and the second term is twice the product of their square roots.

Thus, let x and y represent any two quantities whatever, and we have the general forms of the square as follows:

$$(x+y)^2 = (x+y)(x+y) = x^2+2xy+y^2 \qquad (1)$$
$$(x-y)^2 = (x-y)(x-y) = x^2-2xy+y^2 \qquad (2)$$

The sign of the second term of the square always determines the sign of the second term of the root, y in this particular case.

21. Since x may represent any quantity and y any other quantity, it is evident that *any* trinomial having the form $x^2+2xy+y^2$ or $x^2-2xy+y^2$ is a perfect square.

Rule.—*Extract the square roots of the first and the last term of the trinomial, and connect the results by the sign of the second term.*

EXAMPLE 1.—Factor $x^2 + 6xy + 9y^2$.

SOLUTION.—First see if the trinomial has the form stated in Art **20**. The first and the last terms are seen to be perfect squares, and their roots to be x and $3y$. The second term is also twice the product of the roots x and $3y$, and, since it has the plus sign, the binomial root must be $x + 3y$. Hence, the given expression is a square of the form $x^2 + 2xy + y^2$, and

$$x^2 + 6xy + 9y^2 = (x+3y)(x+3y) = (x+3y)^2 \quad \text{Ans.}$$

EXAMPLE 2.—Factor $16m^4 + 9n^6 - 24m^2n^3$.

SOLUTION.—The first term of the expression is a perfect square, but the last term is not. Inspecting the second term, it is found to be the square of $3n^3$, and the third term to be twice the product of $3n^3$ and the square root, $4m^2$, of the first term. Arranging the trinomial so that the first and the last term are perfect squares, it becomes $16m^4$

$-24m^2n^3+9n^6$ (a square of the form $x^2-2xy+y^2$); hence, $16m^4+9n^6-24m^2n^3 = 16m^4-24m^2n^3+9n^6 = (4m^2-3n^3)(4m^2-3n^3) = (4m^2-3n^3)^2$. Ans.

EXAMPLE 3 —Factor $4x^2+x^2y^2+2x^2y$.

SOLUTION —Arranging the trinomial so that the first term and the last term are perfect squares, it becomes $4x^2+2x^2y+x^2y^2$ Now, although the first and the last term are perfect squares with roots $2x$ and xy, respectively, the second term is only equal to the product of the roots, hence, the trinomial is *not* a perfect square, and can only be factored by Case I Each term contains x^2, and

$$4x^2+x^2y^2+2x^2y = x^2(4+y^2+2y) \quad \text{Ans.}$$

22. *When two of the terms of a trinomial are perfect squares, and have like signs, and the other term is twice the product of their roots, the trinomial is a perfect square.*

Compare this statement with Art. **20.** Thus, $2ab-a^2-b^2$, if divided by -1, becomes $-2ab+a^2+b^2 = a^2-2ab+b^2$; hence, $2ab-a^2-b^2 = -(a^2-2ab+b^2) = -(a-b)^2$.

EXAMPLE 1 —Factor $4pq-4p^2-q^2$.

SOLUTION.—Dividing first by -1, we have $-4pq+4p^2+q^2 = 4p^2-4pq+q^2 = (2p-q)^2$.
Hence, $4pq-4p^2-q^2 = -(4p^2-4pq+q^2) = -(2p-q)^2$ Ans

EXAMPLE 2.—Factor $16r^2s^2+16r^4+4s^4$.

SOLUTION.—The expression contains three squares, but, by careful inspection, we see that the first term is also twice the product of the square roots of the other two.
Thus, $16r^2s^2+16r^4+4s^4 = 16r^4+16r^2s^2+4s^4 = (4r^2+2s^2)^2$ Ans

23. The two formulas given in Art **20** are also used to write out the square of the sum or the difference of two quantities in place of actually performing the multiplication These are expressed in words as follows:

The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first and the second, plus the square of the second.

The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first and the second, plus the square of the second.

EXAMPLE 1.—What is the value of $(2ax + 3b^2)^2$?

SOLUTION.—Here x in formula **1**, Art **20**, equals $2ax$ and $y = 3b^2$; hence, $x^2 = 4a^2x^2$, $2xy = 2 \times 2ax \times 3b^2 = 12ab^2x$, and $y^2 = 9b^4$. Therefore, $(2ax + 3b^2)^2 = 4a^2x^2 + 12ab^2x + 9b^4$ Ans.

EXAMPLE 2.—What is the square of $x^2 - 2ax + a^2$?

SOLUTION.— $x^2 - 2ax + a^2 = (x^2 - 2ax) + a^2$. Now treating the two terms in parenthesis as a single term, let x in formula **2** equal $x^2 - 2ax$ and y equal a^2. $x^2 = (x^2 - 2ax)^2 = x^4 - 4ax^3 + 4a^2x^2$ (applying formula **2**); $2xy = 2 \times (x^2 - 2ax) \times a^2 = 2a^2x^2 - 4a^3x$, and $y^2 = a^4$. Therefore,

$$(x^2 - 2ax + a^2)^2 = x^4 - 4ax^3 + 4a^2x^2 + 2a^2x^2 - 4a^3x + a^4 = x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4 \text{ Ans.}$$

The same result will be obtained if the second and third terms are included in parenthesis and treated as one term. Thus, $x^2 - 2ax + a^2 = x^2 - (2ax - a^2)$. See Art **53**, Part 1. Let x in formula **2** equal x^2 and $y = 2ax - a^2$. Then, $x^2 = (x^2)^2 = x^4$; $2xy = 2 \times x^2 \times (2ax - a^2) = 4ax^3 - 2a^2x^2$, and $y^2 = (2ax - a^2)^2 = 4a^2x^2 - 4a^3x + a^4$. Therefore, $(x^2 - 2ax + a^2)^2 = x^4 - (4ax^3 - 2a^2x^2) + 4a^2x^2 - 4a^3x + a^4 = x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4$ Ans.

24. After a little practice, simple expansions like those in the last two examples can be written directly. Formulas **1** and **2** are very important and should be thoroughly memorized.

EXAMPLES FOR PRACTICE

Factor the following trinomials:

1. $x^2 - 16x + 64$. Ans. $(x - 8)^2$
2. $n^6 - 26n^3 + 169$. Ans. $(n^3 - 13)^2$
3. $25x^2 + 70xyz + 49y^2z^2$. Ans. $(5x + 7yz)^2$
4. $16c^2 + b^2 - 8bc$. Ans. $(4c - b)^2$
5. $2mx - m^2 - x^2$. Ans. $-(m - x)^2$
6. $a^2b^4c^6 - 2ab^2c^3 + 1$. Ans. $(ab^2c^3 - 1)^2$

Square the following:

7. $m + n$. Ans. $m^2 + 2mn + n^2$
8. $4x + 2$. Ans. $16x^2 + 16x + 4$
9. $3a - 5b$. Ans. $9a^2 - 30ab + 25b^2$
10. Square $2c^2 - c + d$.

NOTE.—First separate $2c^2 - c + d$ into two terms by enclosing $c + d$ in parenthesis, then the expression becomes $2c^2 - (c - d)$, and considering this as a binomial, the square is $4c^4 - 4c^2(c - d) + (c - d)^2$.

$$-4c^2(c - d) = -4c^3 + 4c^2d$$
$$(c - d)^2 = c^2 - 2cd + d^2$$

Adding these results to $4c^4$, the final result is $4c^4 - 4c^3 + 4c^2d + c^2 - 2cd + d^2$. Ans.

CASE III

25. To factor an expression that is the difference between two perfect squares:

Formula,

$$(x+y)(x-y) = x^2 - y^2$$

26. Since x may represent any quantity and y any other quantity, it is evident from the formula that any expression that is the difference between two perfect squares may be factored by the following:

Rule.—*Extract the square roots of the first and last terms Add these roots for the first factor, and subtract the second from the first for the second factor.*

EXAMPLE 1 —Factor $9x^8y^6 - 4$.

SOLUTION —The square roots of the first and last terms are $3x^4y^3$ and 2. The sum of these roots is $3x^4y^3 + 2$, and the second subtracted from the first is $3x^4y^3 - 2$.

Hence, $9x^8y^6 - 4 = (3x^4y^3 + 2)(3x^4y^3 - 2)$ Ans.

EXAMPLE 2.—Factor $(a+b)^2 - m^2n^2$.

SOLUTION.—The square roots of the first and last terms are $a + b$ and mn. The sum of these roots is $a + b + mn$, and the second subtracted from the first is $a + b - mn$

Hence,

$$(a+b)^2 - m^2n^2 = (a+b+mn)(a+b-mn) \quad \text{Ans.}$$

27. The formula in Art. **25** is also generally used to write out the product of the sum and difference of two quantities, without actually performing the multiplication. The formula is stated in words as follows:

The product of the sum and difference of two quantities is equal to the difference of their squares.

EXAMPLE 1.—Expand $(x^2+3)(x^2-3)$.

SOLUTION —The square of the first term is x^4, and of the second, 9.

Hence, $(x^2+3)(x^2-3) = x^4 - 9$ Ans.

EXAMPLE 2.—Expand $(ax^2 + bx - 1)(ax^2 + bx + 1)$.

SOLUTION.— $ax^2 + bx - 1 = (ax^2 + bx) - 1$, $ax^2 + bx + 1 = (ax^2 + bx) + 1$, their product is, therefore, $(ax^2 + bx)^2 - 1$, the two terms

in parenthesis being treated as one term If desired to expand further, the expression becomes, applying formula **1**, Art. **20**, $a^2x^4+2abx^3+b^2x^2-1$. Ans.

Example 8 —Factor the expression obtained for the answer to the last example.

Solution —On examining the expression $a^2x^4+2abx^3+b^2x^2-1$, it is seen that the first three terms are composed of descending powers of x with but two literal coefficients Hence, these terms are separated from the last term, -1, for investigation, thus obtaining $(a^2x^4+2abx^3+b^2x^2)-1$, which equals $x^2(a^2x^2+2abx+b^2)-1$. The expression in parenthesis is evidently the square of $ax+b$, hence, the expression becomes $x^2(ax+b)^2-1$ In this last expression the first term is a perfect square, and since the second term is also a perfect square, $x^2(ax+b)^2-1=[x(ax+b)+1][x(ax+b)-1]=(ax^2+bx+1)(ax^2+bx-1)$. Ans.

EXAMPLES FOR PRACTICE

Factor the following expressions:

1. a^2-16. Ans. $(a+4)(a-4)$
2. a^2-49c^8. Ans. $(a+7c^4)(a-7c^4)$
3. $81x^6y^4-1$. Ans. $(9x^3y^2+1)(9x^3y^2-1)$
4. $(ax+by)^2-1$. Ans $(ax+by+1)(ax+by-1)$
5. $25x^4y^2-(bx+1)^2$. Ans. $[5x^2y+(bx+1)][5x^2y-(bx+1)]$ $=(5x^2y+bx+1)(5x^2y-bx-1)$
6. $1-169x^2y^4z^6$. Ans. $(1+13xy^2z^3)(1-13xy^2z^3)$

Expand the following:

7. $(m+1)(m-1)$. Ans. m^2-1
8. $(x^2+y^2)(x^2-y^2)$. Ans. x^4-y^4
9. $(4a+4b^2)(4a-4b^2)$. Ans. $16a^2-16b^4$

28. In example 5, the expression $(bx+1)^2$ should be regarded as a single term; in fact, any number of terms may be regarded as a single term by enclosing them in parenthesis and operating on them as though they were a single letter.

When solving any examples requiring the application of the rules in Art. **26** or **29**, first ascertain if the numerical coefficients of the two terms are perfect squares or perfect cubes; if not, there is no use of examining further.

CASE IV

29. To factor an expression that is the sum or difference of two perfect cubes.

Letting x represent one quantity and y some other quantity, the sum and the difference of two perfect cubes will be represented by x^3+y^3 and x^3-y^3 By actual division it may be shown that

$$(x^3+y^3)\div(x+y) = x^2-xy+y^2 \qquad (1)$$

and $$(x^3-y^3)\div(x-y) = x^2+xy+y^2 \qquad (2)$$

Hence, any expression that is the sum or difference of two perfect cubes may be factored as follows:

Rule.—*Extract the cube root of each term. Connect the results by the sign of the second term for the first factor, and obtain the second factor by division.*

It is to be noticed that the second factor will not be a perfect square, because its second term will not be twice the product of the square roots of the other two.

EXAMPLE.—Factor $8x^6-27y^9$.

SOLUTION.—The cube root of the first term is $2x^2$, and of the second term $3y^3$; the sign of the second term is minus Consequently, the first factor is $2x^2-3y^3$. The second factor is found by division to be $4x^4+6x^2y^3+9y^6$. Hence, the factors are $2x^2-3y^3$ and $4x^4+6x^2y^3+9y^6$. Ans.

EXAMPLES FOR PRACTICE

Factor the following expressions·

1. x^3-8y^3. Ans $(x-2y)(x^2+2xy+4y^2)$
2. m^3+64n^6. Ans. $(m+4n^2)(m^2-4mn^2+16n^4)$
3. $27a^3-8x^3$. Ans. $(3a-2x)(9a^2+6ax+4x^2)$
4. $1{,}000-27a^6b^3$. Ans. $(10-3a^2b)(100+30a^2b+9a^4b^2)$
5. $1+729m^{12}n^{15}$. Ans. $(1+9m^4n^5)(1-9m^4n^5+81m^8n^{10})$
6. $512a^3-64b^3$. Ans. $(8a-4b)(64a^2+32ab+16b^2)$

CASE V

30. Sometimes expressions may be resolved into two or more factors by the application of more than one of the given rules. The student should make himself so familiar with the first four cases that he will be able to determine readily when any of them may be applied

When Case I is to be used in connection with other cases, it should be applied first.

EXAMPLE 1.—Factor $3mx^2y^3 - 12my^7$.

SOLUTION.—By Case I, $3mx^2y^3 - 12my^7 = 3my^3(x^2 - 4y^4)$. Factoring the expression in the parenthesis by Case III, $x^2 - 4y^4 = (x + 2y^2)(x - 2y^2)$

Hence, $3mx^2y^3 - 12my^7 = 3my^3(x + 2y^2)(x - 2y^2)$ Ans.

EXAMPLE 2—Factor $80a^2x^2 - 40ax^2 + 5x^2$.

SOLUTION—By Case I, $80a^2x^2 - 40ax^2 + 5x^2 = 5x^2(16a^2 - 8a + 1)$. Factoring the expression in the parenthesis by Case II, $16a^2 - 8a + 1 = (4a - 1)^2$.

Hence, $80a^2x^2 - 40ax^2 + 5x^2 = 5x^2(4a - 1)^2$ Ans.

EXAMPLE 3—Factor $2mn + 1 - m^2 - n^2$

SOLUTION.—Arrange the expression as follows: $1 - m^2 + 2mn - n^2 = 1 - (m^2 - 2mn + n^2)$. By Case II, this equals $1 - (m - n)^2$. By Case III, this equals $[1 + (m - n)][1 - (m - n)] = (1 + m - n)(1 - m + n)$ Ans.

EXAMPLE 4.—Factor $a^6 - b^6$.

SOLUTION.—By Case III, $a^6 - b^6 = (a^3 + b^3)(a^3 - b^3)$. By Case IV, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$, and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.
Hence,

$$a^6 - b^6 = (a + b)(a - b)(a^2 - ab + b^2)(a^2 + ab + b^2) \quad \text{Ans.}$$

EXAMPLE 5—Factor $4a^2 + x^4 - c^2 + 2cd + 4ax^2 - d^2$.

SOLUTION.—This may be arranged as follows $4a^2 + 4ax^2 + x^4 - c^2 + 2cd - d^2 = 4a^2 + 4ax^2 + x^4 - (c^2 - 2cd + d^2)$.

By Case II, this equals $(2a + x^2)^2 - (c - d)^2$. Hence, by Case III, $4a^2 + x^4 - c^2 + 2cd + 4ax^2 - d^2 = (2a + x^2 + c - d)(2a + x^2 - c + d)$ Ans.

EXAMPLE 6.—Factor $ac - bc + ad - bd$.

SOLUTION—We observe that, if the first two and last two terms be factored by Case I, they will each show the same binomial factor, $a - b$. Thus, $ac - bc + ad - bd = (ac - bc) + (ad - bd) = c(a - b) + d(a - b)$. Applying Case I again, we have (dividing by $a - b$) for the factors $(a - b)(c + d)$ Ans.

EXAMPLE 7.—Factor $x^2 + ax - bx - ab$.

SOLUTION.—This example is like the last. Here, $x^2 + ax - bx - ab = (x^2 + ax) - (bx + ab) = x(x + a) - b(x + a) = (x + a)(x - b)$. Ans.

31. When factoring polynomials which come under Case V, first ascertain whether there is a monomial factor in the expression. If there is one, divide it out and reserve it. If the remaining terms cannot apparently be factored by Cases II, III, and IV, endeavor to so arrange the various terms that they may be factored by application of some of the preceding rules. No fixed rules can be given that will cover all the different expressions which fall under Case V, and the results depend entirely on the ingenuity of the student, who must have considerable practice before he can factor polynomials successfully. It is important, however, that he should have some knowledge of the process. The explanations to the following examples are more full than those given above, and will probably afford some assistance to the understanding of the solutions given under Case V.

EXAMPLE 1.—Factor $ax^6 - ay^6 + b^2x^6 - b^2y^6$.

SOLUTION.—It is readily seen that a is a factor of the first two terms, and b^2 a factor of the last two. Enclosing the first two and last two terms in parentheses, the polynomial becomes $(ax^6 - ay^6) + (b^2x^6 - b^2y^6)$, which of course equals $a(x^6 - y^6) + b^2(x^6 - y^6)$. It is now seen that both terms of this binomial have the common factor $(x^6 - y^6)$. Dividing it out, the quotient is $a + b^2$. Hence, the required factors are $(a + b^2)$ and $(x^6 - y^6)$. But, since x^6 and y^6 are perfect squares, the quantity $x^6 - y^6$ may be factored by Case III. Thus, $x^6 - y^6 = (x^3 + y^3)(x^3 - y^3)$. Both of the factors last obtained may be factored by Case IV. Thus, $x^3 + y^3 = (x^2 - xy + y^2)(x + y)$ and $x^3 - y^3 = (x^2 + xy + y^2)(x - y)$. Therefore, since it is impossible to factor any further, $ax^6 - ay^6 + b^2x^6 - b^2y^6 = (a + b^2)(x^2 - xy + y^2)(x^2 + xy + y^2)(x + y)(x - y)$. Ans.

EXAMPLE 2.—Factor $4 - 9m^2 - n^2 + 6mn$.

SOLUTION.—Apparently, none of the rules will apply here; hence, the chief dependence must be placed on the proper arrangement of the terms. Noticing that the terms $9m^2$ and n^2 are both perfect squares and have like signs, and that the term $6mn$ is twice the product of the square roots of $9m^2$ and n^2, the last three terms are enclosed in parenthesis, and the expression becomes $4 - (9m^2 + n^2 - 6mn)$.

The second term of this binomial is a perfect square, according to Art. **20,** and the binomial may be written $4-(3m-n)^2$, since $(3m-n)^2 = 9m^2 - 6mn + n^2$ The binomial $4-(3m-n)^2$ may now be factored by Case III, since both terms are perfect squares. Therefore, $4-(3m-n)^2 =$

$$[2+(3m-n)][2-(3m-n)] = (2+3m-n)(2-3m+n) \quad \text{Ans.}$$

If the student will carefully study the following Examples for Practice in connection with the foregoing, he should experience no great difficulty in factoring. Until he has become accustomed to factoring, the student should prove his work by multiplying the factors together, and comparing the result with the original expression.

EXAMPLES FOR PRACTICE

Factor the following expressions:

1. $x^4 - y^4$. Apply Case III twice. Ans $(x^2+y^2)(x+y)(x-y)$
2. $3abx^2 + 3ay^2b + 6axyb$. Apply Cases I and II. Ans. $3ab(x+y)^2$
3. $a^4b^2 - ab^5$. Apply Cases I and IV. Ans. $ab^2(a-b)(a^2+ab+b^2)$
4. $2bc - b^2 - c^2 + 4$. Ans. $(2+b-c)(2-b+c)$
5. $16m^2 - 25d^4 + 4n^2 + 16mn$. Ans. $(4m+2n+5d^2)(4m+2n-5d^2)$
6. $y^2 - ay + by - ab$. Ans. $(y-a)(y+b)$
7. $c^2 - 1 + 4x - 4x^2 - 2cd^2 + d^4$. Apply Cases II and III after arranging the terms as follows: $(c^2 - 2cd^2 + d^4) - (4x^2 - 4x + 1)$.
Ans. $(c-d^2+2x-1)(c-d^2-2x+1)$
8. $a^2 - x^2 - 1 + 2x$. Apply Cases II and III.
Ans $(a+1-x)(a-1+x)$
9. $4b^3 - 16ab^3 + 16a^2b^3$. Apply Cases I and II. Ans. $4b^3(1-2a)^2$
10. $x^8 - m^8$. Ans. $(x^4+m^4)(x^2+m^2)(x+m)(x-m)$

CASE VI

32. Expressions of the form $x^n \pm y^n$ frequently occur, in which n is an integer (whole number). The sign $\pm$ is read *plus or minus*, and means that either sign may be used. *One of the factors will be $x+y$, when n is an* **even** *number (2, 4, 6, etc.) and the connecting sign is $-$, or when n is an odd number (3, 5, 7, etc.) and the connecting sign is $+$. When the connecting sign is $-$, $x-y$ is* **always** *a factor.*

$x^n + y^n$ cannot be factored when n is even unless n is exactly divisible by some odd number that is greater than 1.

Thus, $x^4 - y^4$ may be divided by $x + y$, and also by $x - y$; $x^4 + y^4$ cannot be factored; $x^5 + y^5$ may be divided by $x + y$; $x^5 - y^5$ may be divided by $x - y$. $x^6 + y^6$ can be factored, since it equals $x^{2\times3} + y^{2\times3}$; it is divisible by $x^2 + y^2$. Since 1 with any exponent equals 1 (that is $1^3 = 1$, $1^8 = 1$, $1^{10} = 1$, etc.), any root of 1 will also equal 1. Therefore, in the above expressions, 1 may be substituted for either x or y. Thus, $x^4 - 1$ is divisible by $x + 1$ and $x - 1$; $1 - y^4$ is divisible by $1 + y$ and $1 - y$, etc.

FRACTIONS

REDUCTION OF FRACTIONS

DEFINITIONS

33. A **fraction,** in algebra, is considered as an expression indicating division. The sign $\div$ is seldom used, it being more convenient to write the dividend, or quantity to be divided, above a horizontal line, with the divisor below it, in the form of a fraction

Thus, the fraction $\frac{a+b}{c-d}$ means that $a + b$ is to be divided by $c - d$, and is the same as $(a + b) \div (c - d)$. It is read "$a + b$ divided by $c - d$" or "$a + b$ *over* $c - d$." All fractions are read in this way in algebra, except simple numerical fractions, as $\frac{1}{2}$, $\frac{25}{32}$, etc., which are read as in arithmetic.

34. The quantities above and below the line are called the **numerator** and the **denominator,** respectively, as in the case of numerical fractions. They are known as the **terms** of a fraction.

35. Since dividing any quantity by 1 does not change its value, we may write any quantity as a fraction by making

the quantity itself the numerator and 1 the denominator. Thus, $7x^2y$ may be written $\frac{7x^2y}{1}$ and not be altered in value.

36. The **three signs of a fraction** are: the sign before the dividing line, which affects the entire fraction; the sign of the numerator; and the sign of the denominator. When any one of these signs is omitted, it is understood to be plus. *Any two signs of a fraction may be changed without altering its value, but if any one, or all three, be changed, the value of the fraction will be changed from + to − or from − to +.*

When either the numerator or the denominator has more than one term, it should be enclosed in a parenthesis when performing operations affecting it as a whole. The parenthesis may be removed after the operations are completed.

Take the fraction $-\frac{a-b}{c-d}$; placing numerator and denominator in parentheses, we have $-\frac{(a-b)}{(c-d)}$. The signs of the numerator and denominator are each + and that of the fraction −.

Let the quotient of $(a-b) \div (c-d) = q$; then,

$$-\frac{+(a-b)}{+(c-d)} = -(+q) = -q \qquad (1)$$

$$-\frac{-(a-b)}{-(c-d)} = -(+q) = -q \qquad (2)$$

$$-\frac{+(a-b)}{-(c-d)} = -(-q) = +q \qquad (3)$$

$$-\frac{-(a-b)}{+(c-d)} = -(-q) = +q \qquad (4)$$

$$+\frac{+(a-b)}{+(c-d)} = +(+q) = +q \qquad (5)$$

$$+\frac{-(a-b)}{-(c-d)} = +(+q) = +q \qquad (6)$$

$$+\frac{+(a-b)}{-(c-d)} = +(-q) = -q \qquad (7)$$

$$+\frac{-(a-b)}{+(c-d)} = +(-q) = -q \qquad (8)$$

Taking equation (1) as the standard, the sign of the quotient is minus. In equations (2), (7), and (8) *two* of the signs were changed, but the sign of the quotient remained unchanged In equations (3), (4), and (5) only *one* sign was changed, with the result that the sign of the quotient was changed from minus to plus. In equation (6) all *three* signs were changed, resulting in a change in the sign of the quotient from minus to plus.

PRINCIPLES USED IN REDUCTION

37. To *reduce* a fraction is to change its form without changing its value. Thus, $\frac{10x}{5}$ and $\frac{20x}{10}$ have different forms, but like values, since $10x \div 5$ and $20x \div 10$ are each equal to $2x$

The terms of a fraction may both be multiplied, or may both be divided by the same quantity without changing their value.

38. To reduce a fraction to its simplest form:

Rule.—*Resolve each term into its factors, and cancel those factors that appear in both terms.*

39. *In performing all operations on fractions, the student must learn to use a polynomial factor as a single quantity, like a monomial factor.*

This is illustrated in the following examples, where there are polynomial factors in both numerator and denominator that can be canceled.

EXAMPLE 1.—Reduce $\frac{x^2+2xy+y^2}{x^2-y^2}$ to its simplest form.

SOLUTION.—Factoring both numerator and denominator

$$\frac{x^2+2xy+y^2}{x^2-y^2} = \frac{(x+y)(x+y)}{(x+y)(x-y)}$$

Canceling the common factor $x+y$ from both gives, as the result,

$$\frac{\cancel{(x+y)}(x+y)}{\cancel{(x+y)}(x-y)} = \frac{x+y}{x-y}. \quad \text{Ans.}$$

EXAMPLE 2.—Reduce $\frac{3x^5 - 6x^4y}{6x^2y^2 - 12xy^3}$ to its simplest form.

SOLUTION.— $\frac{3x^5 - 6x^4y}{6x^2y^2 - 12xy^3} = \frac{3x^4(x-2y)}{6xy^3(x-2y)}$, when factored.

Canceling the common factors, the result is

$$\frac{\overset{x^3}{\cancel{3}\cancel{x^4}}\cancel{(x-2y)}}{\underset{2}{\cancel{6}}\cancel{x}y^3\cancel{(x-2y)}} = \frac{x^3}{2y^3}.\quad \text{Ans.}$$

40. Sometimes the whole numerator is contained in the denominator, or the denominator in the numerator. The numerator or denominator will then reduce to the number 1.

EXAMPLE 1.—Reduce $\frac{b + 3c^2}{2b^2 + 6bc^2}$ to its simplest form.

SOLUTION.— $\frac{b+3c^2}{2b^2+6bc^2} = \frac{\cancel{b+3c^2}}{2b(\cancel{b+3c^2})} = \frac{1}{2b}.$ Ans.

EXAMPLE 2.—Reduce $\frac{x^6 - 1}{x^3 - 1}$ to its simplest form.

SOLUTION.— $\frac{x^6-1}{x^3-1} = \frac{(x^3+1)\cancel{(x^3-1)}}{\cancel{x^3-1}} = \frac{x^3+1}{1} = x^3+1.$ Ans.
(Art. **35.**)

41. From the last example it will be seen that division may sometimes be performed by cancelation. Thus, $\frac{x^6 - 1}{x^3 - 1}$ means $(x^6 - 1) \div (x^3 - 1)$, and the divisor $x^3 - 1$ canceled from the dividend $x^6 - 1$ gives the quotient $x^3 + 1$.

A factor must be common to each term of the numerator and to each term of the denominator in order to be canceled. Thus, the factor x cannot be canceled from $\frac{3ax}{x + 4m}$ because it is not common to both terms of the denominator.

EXAMPLES FOR PRACTICE

Reduce the following to their simplest form:

1. $\frac{3a + 3b}{a^2 - b^2}$. Ans. $\frac{3}{a-b}$

2. $\frac{x^4 - y^4}{x^2 - y^2}$. Ans. $x^2 + y^2$

3. $\dfrac{54a^3b^5c^2}{72a^2b^2c}$. Ans. $\dfrac{3ab^3c}{4}$

4. $\dfrac{12a^2x^3}{36a^3x^5}$. Ans. $\dfrac{1}{3ax^2}$

5. $\dfrac{n^3-2n^2}{n^2-4n+4}$. Ans. $\dfrac{n^2}{n-2}$

REDUCING FRACTIONS TO A COMMON DENOMINATOR

42. When fractions are to be added or subtracted, it is necessary to so reduce them that all the denominators will be alike. This is called **reducing them to a common denominator.**

43. To reduce fractions to a common denominator:

Rule.—*Resolve each denominator into its factors.*

Take each factor the greatest number of times it occurs in any denominator, and find the product of these factors.

Divide this product by each of the denominators. Multiply the corresponding numerators by these quotients, for new numerators. Write each new numerator with the common denominator beneath it.

EXAMPLE.—Reduce $\dfrac{7a}{x+y}$, $\dfrac{3ab}{x^2-y^2}$, and $\dfrac{2b}{(x+y)^2}$ to a common denominator.

SOLUTION.—Factoring the denominators, $x+y$ is not factorable. $x^2-y^2 = (x+y)(x-y)$, and $(x+y)^2 = (x+y)(x+y)$. Now here are two separate factors, $x+y$ and $x-y$, of which $x+y$ occurs twice in $(x+y)^2$ Hence, the common denominator is $(x+y)(x+y)(x-y) = (x+y)^2(x-y)$. Dividing this product by $x+y$, the quotient is $(x+y)(x-y) = x^2-y^2$. Hence, the first new numerator is $7a(x^2-y^2)$ and the new fraction is $\dfrac{7a(x^2-y^2)}{(x+y)^2(x-y)}$. Similarly, $\dfrac{3ab}{x^2-y^2}$ becomes $\dfrac{3ab(x+y)}{(x+y)^2(x-y)}$, and $\dfrac{2b}{(x+y)^2}$ becomes $\dfrac{2b(x-y)}{(x+y)^2(x-y)}$. Ans.

The student should note that this denominator can be written in several different ways, and he should not become confused if his work does not always agree with the answer.

Besides $(x+y)(x+y)(x-y)$ and $(x+y)^2(x-y)$, it may be written $(x^2-y^2)(x+y)$, $(x^2+2xy+y^2)(x-y)$, or $x^3 + x^2y - xy^2 - y^3$ These five expressions have exactly the same value. The student should prove this statement by substituting numbers for x and y.

EXAMPLES FOR PRACTICE

Reduce the following to common denominators:

1. $\frac{3yz}{2x}$, $\frac{4xz}{3y}$, and $\frac{5xy}{4z}$. Ans. $\frac{18y^2z^2}{12xyz}$, $\frac{16x^2z^2}{12xyz}$, and $\frac{15x^2y^2}{12xyz}$

2. $\frac{x^2y}{10}$, $\frac{xyz}{15}$, and $\frac{7yz^2}{30}$. Ans $\frac{3x^2y}{30}$, $\frac{2xyz}{30}$, and $\frac{7yz^2}{30}$

3. $\frac{2}{a^3x^3}$, $\frac{3}{ax^3}$, and $\frac{4}{a^2x}$. Ans. $\frac{2}{a^3x^3}$, $\frac{3a^2}{a^3x^3}$, and $\frac{4ax^2}{a^3x^3}$

4. $\frac{m+n}{m-n}$, and $\frac{m-n}{m+n}$ Ans. $\frac{m^2+2mn+n^2}{m^2-n^2}$, and $\frac{m^2-2mn+n^2}{m^2-n^2}$

5. $\frac{2}{x}$, $\frac{3}{2x-1}$, and $\frac{2x-1}{4x^2-1}$.

Ans. $\frac{2(4x^2-1)}{x(4x^2-1)}$, $\frac{3x(2x+1)}{x(4x^2-1)}$, and $\frac{x(2x-1)}{x(4x^2-1)}$

OPERATIONS WITH FRACTIONS

ADDITION AND SUBTRACTION OF FRACTIONS

44. To add or subtract fractions:

Rule.—*Reduce the fractions, if necessary, to a common denominator. Add or subtract the numerators, and write the result over the common denominator.*

Example 1.—Find the sum of $\frac{2a-b}{5}$ and $\frac{a+b}{4}$.

Solution — $\frac{2a-b}{5}$ and $\frac{a+b}{4}$, reduced to a common denominator, become $\frac{4(2a-b)}{20}$ and $\frac{5(a+b)}{20}$, which are equal, respectively, to $\frac{8a-4b}{20}$ and $\frac{5a+5b}{20}$. Adding the numerators, the result is $8a-4b$

$+ 5a + 5b = 13a + b$, which written over the common denominator gives as the sum, $\frac{13a + b}{20}$ The work is written as follows:

$$\frac{2a - b}{5} + \frac{a + b}{4} = \frac{8a - 4b}{20} + \frac{5a + 5b}{20}$$

$$= \frac{8a - 4b + 5a + 5b}{20} = \frac{13a + b}{20} \quad \text{Ans.}$$

EXAMPLE 2.—Subtract $\frac{6b - 2}{3b}$ from $\frac{4a - 1}{2a}$.

SOLUTION —Reducing the fractions to a common denominator, $\frac{4a - 1}{2a} - \frac{6b - 2}{3b} = \frac{12ab - 3b}{6ab} - \frac{12ab - 4a}{6ab}$. Subtracting the second numerator from the first, and writing the result over the common denominator, $\frac{12ab - 3b}{6ab} - \frac{12ab - 4a}{6ab} = \frac{(12ab - 3b) - (12ab - 4a)}{6ab}$ $= \frac{12ab - 3b - 12ab + 4a}{6ab}$, with the parentheses removed. Combining like terms in the numerator gives as the result $\frac{4a - 3b}{6ab}$. Ans.

45. *If, as in the example just given, the numerator of the fraction to be subtracted has more than one term*, care must be taken *to change the sign of every term before combining*. It will usually be convenient to enclose the whole numerator in a parenthesis before combining. The parenthesis may then be removed by the principles of Arts. **49, 50,** and **51,** Part 1.

EXAMPLE 1.—Simplify $\frac{x^3}{x - 1} - \frac{x^2}{x + 1} - \frac{x}{x - 1} + \frac{1}{x + 1}$.

SOLUTION.—Reducing to the common denominator $x^2 - 1$,

$$\frac{x^3}{x - 1} - \frac{x^2}{x + 1} - \frac{x}{x - 1} + \frac{1}{x + 1} = \frac{x^4 + x^3}{x^2 - 1} - \frac{x^3 - x^2}{x^2 - 1} - \frac{x^2 + x}{x^2 - 1} + \frac{x - 1}{x^2 - 1}$$

Adding or subtracting the numerators as required,

$$\frac{(x^4 + x^3) - (x^3 - x^2) - (x^2 + x) + (x - 1)}{x^2 - 1}$$

which, with the parentheses removed,

$$= \frac{x^4 + x^3 - x^3 + x^2 - x^2 - x + x - 1}{x^2 - 1}$$

Combining like terms, the result is

$$\frac{x^4-1}{x^2-1} = x^2+1 \quad \text{Ans.}$$

EXAMPLE 2.—Simplify $\frac{1}{(x-2)^2}+\frac{1}{2-x}$.

SOLUTION.—If the denominator of the second fraction were written $x-2$ instead of $2-x$, $(x-2)^2$ would be the common denominator. By Art **36**, the signs of the denominator and the sign before the fraction $\frac{1}{2-x}$ may be changed, giving $-\frac{1}{-2+x}=-\frac{1}{x-2}$. (Art **24**, Part 1) Hence, $\frac{1}{(x-2)^2}+\frac{1}{2-x}=\frac{1}{(x-2)^2}-\frac{1}{x-2}$, which, when reduced to a common denominator,

$$=\frac{1}{(x-2)^2}-\frac{x-2}{(x-2)^2}=\frac{1-(x-2)}{(x-2)^2}=\frac{1-x+2}{(x-2)^2}=\frac{3-x}{(x-2)^2} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Simplify the following by reducing to simple fractions:

1. $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}$. Ans. $\frac{47x}{60}$

2. $\frac{4x-3}{5}+\frac{7x+1}{8}+\frac{3x}{2}$. Ans. $\frac{189x-8}{80}$

3. $\frac{1}{x-y}-\frac{1}{x^2-y^2}$. Ans. $\frac{x+y-1}{x^2-y^2}$

4. $\frac{a^2+b^2}{2}-\frac{(a+b)^2}{4}$. Ans $\frac{2(a^2+b^2)-(a^2+2ab+b^2)}{4}=\frac{(a-b)^2}{4}$, after removing parentheses and combining.

5. $\frac{a^2}{a^2-1}+\frac{a}{a-1}-\frac{a}{a+1}$. Ans. $\frac{a^2+2a}{a^2-1}$

6. $\frac{4m^2+1}{4m^2}-\frac{3m-1}{12m^3}+\frac{1-12n}{12n}$. Ans. $\frac{n+m^3}{12m^3n}$

7 $\frac{y}{(x+y)^2}+\frac{y}{x^2-y^2}-\frac{1}{x+y}$.

Ans $\frac{y(x-y)+y(x+y)-(x^2-y^2)}{(x+y)^2(x-y)}=\frac{2xy-x^2+y^2}{x^3+x^2y-xy^2-y^3}$

8. $\frac{x}{x+1}+\frac{x}{1-x}+\frac{3x}{x^2-1}$. Ans $\frac{x}{x^2-1}$

MULTIPLICATION OF FRACTIONS

46. To multiply fractions:

Rule.—*Multiply the numerators together for the numerator of the product, and the denominators together for the denominator of the product.*

47. Any number of fractions may be multiplied together. The operation may be very much shortened by resolving the terms of the fractions into their factors, and canceling. The product should be reduced to its simplest form.

EXAMPLE 1.—Find the product of $\frac{6a^2}{5}$, $\frac{2ab}{3c}$, and $\frac{2ac}{b^2}$.

SOLUTION.—The product of the numerators is $6a^2 \times 2ab \times 2ac = 24a^4bc$, and of the denominators, $5 \times 3c \times b^2 = 15b^2c$. Writing $24a^4bc$ over $15b^2c$, the product of the fractions is $\frac{24a^4bc}{15b^2c} = \frac{8a^4}{5b}$, when reduced to its lowest terms. The work is written as follows:

$$\frac{\overset{2}{\cancel{6}}a^2}{5}\times\frac{2a\cancel{b}}{\cancel{3}\cancel{c}}\times\frac{2a\cancel{c}}{\underset{b}{\cancel{b^2}}} = \frac{8a^4}{5b}. \quad \text{Ans.}$$

EXAMPLE 2.—Find the product of $\frac{x^2+2x}{(x-1)^2}$, x^2-1, and $\frac{x^2-4x+4}{x^2-4}$.

SOLUTION.—First make x^2-1 a fraction by writing 1 for its denominator, thus, $\frac{x^2-1}{1}$; then, factoring both terms of each fraction,

$$\frac{x^2+2x}{(x-1)^2}\times\frac{x^2-1}{1}\times\frac{x^2-4x+4}{x^2-4}$$

$$= \frac{x\cancel{(x+2)}(x+1)\cancel{(x-1)}\cancel{(x-2)}(x-2)}{\cancel{(x-1)}(x-1)\cancel{(x-2)}\cancel{(x+2)}} = \frac{x(x+1)(x-2)}{x-1}. \quad \text{Ans.}$$

EXAMPLE 3.—Find the product of $\frac{1}{a^2}-\frac{4c^2}{a}$, and $\frac{a^2}{1+2ac}$.

SOLUTION.—Performing the subtraction, $\frac{1}{a^2}-\frac{4c^2}{a} = \frac{1-4a^2c^2}{a^2}$.

$$\text{Multiplying, } \frac{1-4a^2c^2}{a^2}\times\frac{a^2}{1+2ac} = \frac{\overset{1-2ac}{\cancel{(1-2ac)(1+2ac)}}\cancel{a^2}}{\cancel{(1+2ac)}\cancel{a^2}} = 1-2ac. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Multiply the following.

1. $\frac{3a^2bc}{5abc^3}$ by $\frac{10ab^2c}{3abc}$. Ans. $\frac{2ab}{c}$

2. $\frac{5x^2y}{7x}$ by $21xy$. Ans. $15x^2y^2$

Find the product of:

3. $\frac{3x^2y}{4xz^2}$, $\frac{5y^2z}{6xy}$, and $\frac{-12x^2}{2xy^2}$. Ans. $-\frac{15x}{4z}$

4. $\frac{x^2-y^2}{c^3-d^3}$, $\frac{c-d}{(x+y)^2}$, and $\frac{x^3+y^3}{x-y}$. Ans. $\frac{x^2-xy+y^2}{c^2+cd+d^2}$

5. $\frac{4y}{x}-\frac{16}{xy}$, and $\frac{1}{2y+4}$. Ans. $\frac{2y-4}{xy}$

6. $\frac{a+b}{2}+\frac{a-b}{4}$, and $\frac{4}{9a^2+6ab+b^2}$. Ans. $\frac{1}{3a+b}$

DIVISION OF FRACTIONS

48. Division, in fractions, is the reverse of multiplication, and is the process employed when, given one of two fractions and their product, it is required to find the other. For example, it is required to divide $\frac{a}{4}$ by $\frac{1}{2}$. Find such a fraction that, multiplied by $\frac{1}{2}$, will give $\frac{a}{4}$, for the product. This fraction is $\frac{a}{2}$, for $\frac{a}{2}\times\frac{1}{2}=\frac{a}{4}$. Also, $\frac{x}{5}\div\frac{x}{7}=\frac{7}{5}$, since $\frac{7}{5}\times\frac{x}{7}=\frac{x}{5}$. If, in this case, the divisor had been inverted and the fractions multiplied, the result would have been $\frac{x}{5}\times\frac{7}{x}=\frac{7}{5}$.

49. Hence, to divide by a fraction:

Rule.—*Invert the divisor and proceed as in multiplication.*

Example 1.—Divide $\frac{3a^2b}{5x^3y}$ by $\frac{9ab^3}{10x^4y^2}$.

Solution.—The divisor inverted $=\frac{10x^4y^2}{9ab^3}$.

Hence, $\frac{3a^2b}{5x^3y} \div \frac{9ab^3}{10x^4y^2} = \frac{3a^2b}{5x^3y} \times \frac{10x^4y^2}{9ab^3} = \frac{\overset{2a\ \ x\ \ y}{\cancel{3}\times\cancel{10}\cancel{a^2}\cancel{b}\cancel{x^4}\cancel{y^2}}}{\underset{3\ \ b^2}{\cancel{5}\times\cancel{9}\cancel{a}\cancel{b^3}\cancel{x^3}\cancel{y}}} = \frac{2axy}{3b^2}$. Ans.

EXAMPLE 2.—Divide $x^2 + 2x + 1$ by $\frac{x+1}{x-1}$.

SOLUTION.—By Art. 35,

$$\frac{x^2+2x+1}{1} \times \frac{x-1}{x+1} = \frac{\cancel{(x+1)}(x+1)(x-1)}{\cancel{x+1}} = x^2-1 \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Divide the following.

1. $\frac{9x^3 - 3x^4}{24}$ by $\frac{3x}{8}$. Ans. $\frac{3x^2 - x^3}{3}$

2. $\frac{ab - bx}{a + z}$ by $\frac{ac - cx}{a + z}$. Ans $\frac{b}{c}$

3. $\frac{1 - 8b^2 + 16b^4}{1 + 2b}$ by $\frac{1 - 4b^2}{3a}$. Ans $3a(1 - 2b)$

4. $6a^2cd - 6abcd$ by $\frac{6acd}{a^2 + ab + b^2}$. Ans. $a^3 - b^3$

MIXED QUANTITIES AND COMPLEX FRACTIONS

50. An integral expression (see Art. **11**) is one containing neither fractions nor negative exponents. The expression $a^2 + 2ab$ is integral, but the expressions $a^2 + \frac{1}{2ab}$, $2a^{-2}$, $\frac{3}{a^2 + b}$ are not. The expression $2a^{-2}$ is only another way of writing $\frac{2}{a^2}$.

51. The **integral part** of an expression is that part which, if taken by itself, would be an integral expression.

52. A **mixed quantity** is an expression containing both integral and fractional parts, as $2a^2 - \frac{c + d}{4a}$. Considering the integral part, $2a^2$, as a fraction with a denominator 1

(see Art. **35**), a mixed quantity becomes simply the indicated addition or subtraction of two fractions; thus, $2a^2 - \frac{c+d}{4a} = \frac{2a^2}{1} - \frac{c+d}{4a}$.

53. A fraction may be reduced to either an entire or a mixed quantity by dividing the numerator by the denominator, provided the division be possible. It frequently happens that by performing the indicated division, the fraction will be reduced to a simpler form The case of reducing a fraction to an entire quantity was taken up in Art. **40.**

EXAMPLE —Simplify $\frac{4x^2 + 12x - 1}{2x + 3}$.

SOLUTION.—Performing the indicated division,

$$\begin{array}{rl}
2x + 3)4x^2 + 12x - 1 & (2x + 3 - \frac{10}{2x+3} \quad \text{Ans.} \\
4x^2 + 6x & \\
6x - 1 & \\
\underline{6x + 9} & \\
-10 &
\end{array}$$

54. Mixed quantities are frequently more convenient to handle as fractions.

To reduce a mixed quantity to a fraction:

Rule.—*Write the integral part with a denominator 1, and perform the indicated addition or subtraction.*

EXAMPLE.—Reduce $x^2 + xy + y^2 - \frac{b}{x-y}$ to a fraction.

SOLUTION.—

$$x^2 + xy + y^2 - \frac{b}{x-y} = \frac{x^2 + xy + y^2}{1} - \frac{b}{x-y}$$

Subtracting the second fraction from the first gives

$$\frac{(x^2 + xy + y^2)(x - y) - b}{x - y} = \frac{x^3 - y^3 - b}{x - y} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Solve the following·

1. Reduce $\frac{a^2c + b^2}{c}$ to a mixed quantity. Ans. $a^2 + \frac{b^2}{c}$

2. Simplify $\dfrac{x^2 + 4xy + 5y^2 - 3x}{x + 2y}$ Ans. $x + 2y - 3 + \dfrac{y^2 + 6y}{x + 2y}$

3 Reduce $x + 3 - \dfrac{7x + 3}{2x + 1}$ to a fraction. Ans. $\dfrac{2x^2}{2x + 1}$

4 From $3a + \dfrac{a + b}{d}$ subtract $a - \dfrac{a - b}{d}$

Ans. $2a + \dfrac{2a}{d} = \dfrac{2a(d + 1)}{d}$

5. Divide $m + n - \dfrac{2n}{m - n}$ by $m - n - \dfrac{2n}{m + n}$. Ans. $\dfrac{m + n}{m - n}$

SUGGESTION —First reduce the mixed quantities to fractions.

COMPLEX FRACTIONS

55. A **complex fraction** is one that contains fractions in one or both of its terms. Thus, $\dfrac{a + \frac{x}{y}}{a + x}$, $\dfrac{a - b}{\frac{x}{y}}$, and $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ are complex fractions.

56. Complex fractions can be reduced by performing the indicated division; thus, $\dfrac{\frac{5}{8}}{\frac{3}{4}} = \dfrac{5}{8} \div \dfrac{3}{4} = \dfrac{5}{8} \times \dfrac{4}{3} = \dfrac{5}{6}$. A much simpler way is to multiply both terms by the least common denominator of the fractions contained. Thus, $\dfrac{\frac{5}{8} \times 8}{\frac{3}{4} \times 8} = \dfrac{5}{6}$.

57. Hence, to simplify a complex fraction:

Rule.—*Multiply both terms by the common denominator of the fractional parts.*

EXAMPLE 1.—Simplify $\dfrac{\frac{x}{y} - \frac{y}{x}}{\frac{1}{y} - \frac{1}{x}}$.

SOLUTION.—The common denominator of the fractional parts is xy. Multiplying each term by this,

$$\frac{\frac{x}{y} \times xy - \frac{y}{x} \times xy}{\frac{1}{y} \times xy - \frac{1}{x} \times xy} = \frac{x^2 - y^2}{x - y} = x + y \quad \text{Ans.}$$

The multiplication can frequently be performed mentally, without writing the common denominator, at the same time canceling common factors.

EXAMPLE 2.—Simplify $\dfrac{1}{1+\dfrac{a}{1+a+\dfrac{2a^2}{1-a}}}$.

SOLUTION.—This is the case of a complex fraction in which the denominator is itself a complex fraction.

First, consider the part $\dfrac{a}{1+a+\dfrac{2a^2}{1-a}}$.

Multiplying both terms by $1-a$,

$$\frac{a(1-a)}{(1+a)(1-a)+2a^2}=\frac{a-a^2}{1-a^2+2a^2}=\frac{a-a^2}{1+a^2}$$

The fraction thus becomes $\dfrac{1}{1+\dfrac{a-a^2}{1+a^2}}$.

Multiplying both terms by $1+a^2$,

$$\frac{1+a^2}{1+a^2+a-a^2}=\frac{1+a^2}{1+a} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Simplify the following:

1. $\dfrac{\dfrac{3ac^2}{16}}{24}$. Ans. $\dfrac{ac^2}{128}$

2. $\dfrac{1+\dfrac{a}{c}}{c-\dfrac{a^2}{c}}$. Ans. $\dfrac{1}{c-a}$

3. $\dfrac{2\frac{7}{8}}{8-2x+\dfrac{x^2}{8}}$. Ans. $\dfrac{23}{(8-x)^2}$

SUGGESTION.— $2\frac{7}{8}$ means $2+\frac{7}{8}$. Hence, for the numerator multiply 2 by the least common denominator 8, and add 7.

4. $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$. Ans. $\dfrac{4}{3x+3}$

THEORY OF EXPONENTS

58. An **exponent** may be a number, a letter, or a combination of both; it may be integral, fractional, or zero; and it may be positive or negative.

59. When the exponent is a letter, it is called a **literal exponent.** Exponents may involve several letters and terms. The following are examples of exponents: x^3, x^a, x^{3a}, $x^{\frac{2}{3}}$, $x^{-\frac{2a}{3}}$, x^0, x^{2p-q}, etc.

60. It has been shown before (see Art. **70**, Part 1) that any quantity with zero for an exponent is equal to 1. It has also been shown that any quantity having a negative exponent is equal to the reciprocal of the quantity with an equal positive exponent; that is, for example, $x^{-\frac{2a}{3}} = \frac{1}{x^{\frac{2a}{3}}}$. An expression like x^{2p-q} arises from dividing x^{2p} by x^q; thus, $x^{2p} \div x^q = x^{2p-q}$.

61. As it is frequently necessary in algebraic operations to use letters for exponents—as an example see Art. **32**—it also becomes necessary to affect such quantities with exponents and to extract roots. For example, the square of x^n may be written either $(x^n)^2$ or x^{2n}, the latter being a simplified form of the former. An expression of this kind can be best understood by a numerical example. Thus, consider the expression $5^{2\times2\times2}$; this is equivalent to $[(5^2)^2]^2$ and also to 5^8. The advantage of writing it $5^{2\times2\times2}$ is that it indicates how the multiplication may be simplified. For example, instead of multiplying 5 by 5 and this product by 5 and so on until 5 has been used eight times as a factor, simply square 5, then square the product, and then square

the last product; this results in three multiplications instead of seven.

Since $(x^m)^3 = (x^m) \times (x^m) \times (x^m) = x^{3m}$, and $(x^m)^n$ evidently equals x^{nm}, all that is necessary to do in raising an expression like x^m to any power is to multiply the exponent of the given expression by the exponent denoting the power to which the expression is to be raised. An expression like x^{3m} must not be considered as equivalent to $x^m \times x^3$, for the latter expression is equal to x^{m+3}.

62. If an expression like x^{6m} occurs and it is desired to extract, say, the cube root of it, divide the exponent of the expression by the index of the root. Thus, $\sqrt[3]{x^{6m}} = x^{6m \div 3} = x^{2m}$; this is necessarily true since $(x^{2m})^3 = x^{6m}$. But when the exponent is not exactly divisible by the index, the division is indicated by a fraction. Thus, $\sqrt[n]{x^m} = x^{m \div n} = x^{\frac{m}{n}}$; $\sqrt[3]{x^{2a}} = x^{\frac{2a}{3}}$; $\sqrt[4]{x} = x^{\frac{1}{4}}$; $\sqrt[3]{\frac{1}{x^a}} = \frac{\sqrt[3]{1}}{\sqrt[3]{x^a}} = \frac{1}{\sqrt[3]{x^a}}$
$= \frac{1}{x^{\frac{a}{3}}} = x^{-\frac{a}{3}}$, etc.

63. From the foregoing the following rules are evident:

Rule I.—*To raise a monomial to any power, raise the numerical coefficient to the desired power and multiply the exponent of each letter by the exponent denoting the power to which the monomial is to be raised. If the sign of the monomial is plus, or if the sign is minus and the exponent denoting the power is even, the sign of the power will be plus; but if the sign is minus and the exponent denoting the power is odd, the sign of the power will be minus.*

Rule II.—*To extract any root of a monomial, extract the required root of the numerical coefficient and divide the exponent of each letter by the index of the root. If the index is odd, the sign of the root will be the same as the sign of the monomial; but if the index of the root is even and the sign of the monomial is plus, the sign of the root will be* $\pm$, *while if*

the sign of the monomial is minus, the root must be indicated, as it is impossible to extract an even root of a negative quantity.

64. These two rules should be readily understood from what has preceded, but a further discussion of the law of signs will be given. For this purpose consider the two expressions $(\pm x)^n$ and $\sqrt[n]{\pm x}$. These give rise to the following eight cases:

$$\begin{array}{lll}
\text{When } n \text{ is even } (+x)^n = +x^n & & (1)\\
\text{When } n \text{ is odd } (+x)^n = +x^n & & (2)\\
\text{When } n \text{ is even } (-x)^n = +x^n & & (3)\\
\text{When } n \text{ is odd } (-x)^n = -x^n & & (4)\\
\text{When } n \text{ is odd } \sqrt[n]{+x} = +\sqrt[n]{x} & & (5)\\
\text{When } n \text{ is odd } \sqrt[n]{-x} = -\sqrt[n]{x} & & (6)\\
\text{When } n \text{ is even } \sqrt[n]{+x} = \pm\sqrt[n]{x} & & (7)\\
\text{When } n \text{ is even } \sqrt[n]{-x} = \sqrt[n]{-x} & & (8)
\end{array}$$

Cases (1) and (2) are evident, since any positive quantity raised to any power must be positive. Cases (3) and (4) follow from Art. **58**, Part 1. Case (5) is the converse of Case (2), and Case (6) is the converse of Case (4). Case (7) gives an ambiguous result because when n is even $(+x)^n$ and $(-x)^n$ are both equal to $+x^n$, and unless there is something else in the conditions of the problem to determine which sign to use, it is necessary to use the double sign. Case (8) can only be indicated as shown. This can virtually be restricted to $\sqrt{-x}$, in which the index is 2. The square root of a negative quantity is called an **imaginary quantity.** There is no integral or fractional quantity whose square will equal a negative quantity; hence, the square root of such a quantity must be indicated as shown in Case (8).

EXAMPLE 1.—Find the values of the following: $(a^{-1})^{-\frac{1}{2}}$; $(cd^{-2})^{\frac{1}{2}}$; $(x^a)^{-b} + (x^{-a})^{-b}$.

SOLUTION.—In the first, multiplying the exponents, $-1 \times -\frac{1}{2} = \frac{1}{2}$.

Hence, $(a^{-1})^{-\frac{1}{2}} = a^{\frac{1}{2}}$, or $\sqrt{a}$ Ans.

In like manner,

$(c\,d^{-2})^{\frac{5}{2}} = c^{\frac{5}{2}}d^{-5}$, Ans., since $1 \times \frac{5}{2} = \frac{5}{2}$, and $-2 \times \frac{5}{2} = -5$

In the next one,

$$(x^a)^{-b} = x^{-ab} \text{ and } (x^{-a})^{-b} = x^{ab}$$

Dividing, $\quad x^{-ab} \div x^{ab} = x^{-ab-ab} = x^{-2ab}$ Ans.

EXAMPLE 2.—Find the value of $\sqrt[4]{256a^4b^{12}c^8}$.

SOLUTION.—The 4th root of 256 is 4 The exponent of a in the root is $4 \div 4 = 1$, of b, $12 \div 4 = 3$, and of c, $8 \div 4 = 2$ As this is an even root of a positive quantity, the sign should be $\pm$.

Hence, $\quad \sqrt[4]{256a^4b^{12}c^8} = \pm 4ab^3c^2$ Ans.

EXAMPLE 3.—Find the value of $\sqrt[3]{\dfrac{27m^3x^9}{a^9b^6c^{12}}}$.

SOLUTION.— $\sqrt[3]{27m^3x^9} = 3mx^3$, $\sqrt[3]{a^9b^6c^{12}} = a^3b^2c^4$. The quantity is positive, and, as this is an odd root, its sign must be the same, or positive.

Hence, $\quad \sqrt[3]{\dfrac{27m^3x^9}{a^9b^6c^{12}}} = \dfrac{3mx^3}{a^3b^2c^4}$ Ans.

65. Since in $\dfrac{1}{a^2} = a^{-2}$, a^{-2} changes to a^2 when placed in the denominator, we may state the following principle:

A factor may be changed from the numerator to the denominator, or from the denominator to the numerator, if the sign of its exponent be changed.

For example, $\dfrac{n^{-3}}{ab} = \dfrac{1}{abn^3}$; $\dfrac{n}{ab^{-\frac{1}{2}}} = \dfrac{nb^{\frac{1}{2}}}{a}$; $\dfrac{x^{-\frac{2}{3}}}{5y^{-1}} = \dfrac{y}{5x^{\frac{2}{3}}}$, etc.
In the last, the positive exponent 1 of the y is not written.

EXAMPLE.—Express, with positive exponents,

$$a^{-1}b^{-2}c^3 + a^{-2}b^{-\frac{3}{4}}c^{-\frac{1}{3}} + a^3b^{-2}$$

SOLUTION —Since these terms may be taken as fractions, with 1 for the denominators, transfer the letters with negative exponents to the denominators, obtaining

$$a^{-1}b^{-2}c^3 + a^{-2}b^{-\frac{3}{4}}c^{-\frac{1}{3}} + a^3b^{-2} = \frac{c^3}{ab^2} + \frac{1}{a^2b^{\frac{3}{4}}c^{\frac{1}{3}}} + \frac{a^3}{b^2} \text{ Ans.}$$

66. The student must note very carefully that factors of an entire term only can be changed from numerator to denominator, or vice versa, and that when thus changed they become factors of the whole of the other term. Thus, in $\frac{a}{bc^{-2}+d}$, c^{-2} *cannot* be transferred to the numerator by merely changing the sign of the exponent. The exponent may, however, be made positive by multiplying both terms by c^2; thus, $\frac{a \times c^2}{(bc^{-2}+d) \times c^2} = \frac{ac^2}{b+c^2d}$. In $\frac{ac^{-2}}{b+d}$, if we transfer the c^{-2}, it becomes $\frac{a}{c^2(b+d)}$, c^2 becoming a factor of the *entire* denominator.

EXAMPLE 1.—Clear $x^2y^{-2}z^{-1} + \frac{2xy}{y^{-1}-x^3} - \frac{3a^{-1}b^{-2}c^3}{a^2+b}$ of negative exponents.

SOLUTION.—Treat each term of the expression separately $x^2y^{-2}z^{-1} = \frac{x^2y^{-2}z^{-1}}{1}$; changing the factors with negative exponents to the denominator, and at the same time changing the signs of the exponents, the result is $\frac{x^2}{y^2z}$. In $\frac{2xy}{y^{-1}-x^3}$, y^{-1} is not a factor of the whole denominator; hence, multiply both terms of the fraction by the reciprocal of y^{-1} or y; thus, $\frac{2xy \times y}{(y^{-1}-x^3) \times y} = \frac{2xy^2}{1-x^3y}$. In $\frac{3a^{-1}b^{-2}c^3}{a^2+b}$, a^{-1} and b^{-2} are factors of the entire numerator, so we write them as factors of the entire denominator, with the signs of the exponents changed, thus,

$$\frac{3a^{-1}b^{-2}c^3}{a^2+b} = \frac{3c^3}{ab^2(a^2+b)} = \frac{3c^3}{a^3b^2+ab^3}$$

Hence,

$$x^2y^{-2}z^{-1} + \frac{2xy}{y^{-1}-x^3} - \frac{3a^{-1}b^{-2}c^3}{a^2+b} = \frac{x^2}{y^2z} + \frac{2xy^2}{1-x^3y} - \frac{3c^3}{a^3b^2+ab^3} \quad \text{Ans.}$$

EXAMPLE 2.—Solve the following:

$$a^3 \times a^{-1};\ n \times n^{-\frac{1}{2}};\ 2c^{-\frac{2}{3}} \times \frac{1}{-3\sqrt[3]{c^2}};\ c^{\frac{n}{m}} \div c^{\frac{2n}{m}};\ x^2 \div \sqrt[3]{x^2}$$

Write the answers with positive exponents.

SOLUTION.— $a^3 \times a^{-1} = a^{3+(-1)} = a^{3-1} = a^2$ Ans.

$n \times n^{-\frac{1}{2}} = n^{1+(-\frac{1}{2})} = n^{1-\frac{1}{2}} = n^{\frac{1}{2}}$ Ans.

$$2c^{-\frac{2}{3}} \times \frac{1}{-3\sqrt[3]{c^2}} = \frac{2c^{-\frac{2}{3}}}{1} \times -\frac{1}{3\sqrt[3]{c^2}} = \frac{2}{c^{\frac{2}{3}}} \times -\frac{1}{3c^{\frac{2}{3}}} = -\frac{2 \times 1}{c^{\frac{2}{3}} \times 3c^{\frac{2}{3}}}$$

$$= -\frac{2}{3c^{\frac{2}{3}+\frac{2}{3}}} = -\frac{2}{3c^{\frac{4}{3}}} \text{ Ans.}$$

$$c^{\frac{n}{m}} \div c^{\frac{2n}{m}} = c^{\frac{n}{m}-\frac{2n}{m}} = c^{\frac{n-2n}{m}} = c^{\frac{-n}{m}} = c^{-\frac{n}{m}} = \frac{1}{c^{\frac{n}{m}}} \text{ Ans.}$$

$$x^2 \div \sqrt[5]{x^2} = x^2 \div x^{\frac{2}{5}} = x^{2-\frac{2}{5}} = x^{\frac{10}{5}-\frac{2}{5}} = x^{\frac{8}{5}} \text{ Ans.}$$

EXAMPLES FOR PRACTICE

Clear the following of negative exponents:

1. $x^2y^{-2}z^{-\frac{1}{2}}$. Ans. $\frac{x^2}{y^2z^{\frac{1}{2}}}$

2. $3a^{-1}b + \frac{2a}{b^{-3}c^{-1}} + c^{-1}$. Ans. $\frac{3b}{a} + 2ab^3c + \frac{1}{c}$

3. $\frac{4a^{-2}(c+d)}{2c+d}$. Ans. $\frac{4(c+d)}{a^2(2c+d)}$

Express the following without radical signs:

4. $\sqrt[3]{b^{-2}}$ Ans. $(b^{-2})^{\frac{1}{3}}$ or $b^{-\frac{2}{3}}$

5. $4a\sqrt{a^{-1}b^{-3}}$. Ans. $4aa^{-\frac{1}{2}}b^{-\frac{3}{2}} = 4a^{\frac{1}{2}}b^{-\frac{3}{2}}$

Find the values of the following:

6. $m^{\frac{1}{2}} \times m^{-\frac{1}{3}}$. Ans. $m^{\frac{1}{6}}$

7. $2ab^{\frac{1}{2}} \times a^{-\frac{1}{2}}b$. Ans. $2a^{\frac{1}{2}}b^{\frac{3}{2}}$

8. $c^{\frac{n}{2}} \div \sqrt{c^{-n}}$ Ans. c^n

9. $2x^{-2} \div (x^2)^{-\frac{1}{2}}$. Ans. $2x^{-1}$

10. $\left(cd^{-\frac{n}{m}}\right)^{2m} \times \sqrt{d^{4n}}$. Ans. c^{2m}

ELEMENTS OF ALGEBRA

(PART 3)

EQUATIONS

1. As defined in Art. **5**, Part 1, an **equation** is a statement of equality between two expressions, as $x + 6 = 14$.

2. Every equation has two parts, called the **first** and **second members.** The first member is the part on the left of the sign of equality, and the second member the part on the right of that sign. In $x + 6 = 14$, $x + 6$ is the first member, and 14 is the second member.

3. Equations usually consist of **known** and **unknown quantities**; that is, of quantities whose values are given, and of quantities whose values are not given, but are to be found. Thus, in $x + 6 = 14$, 6 and 14 are known quantities, and x is unknown; but since by the statement of the equation, $x + 6$ must equal 14, x must have such a value that when added to 6 the sum will be 14. Hence, the value of x is fixed for this particular case, and in a similar manner the value of a single unknown quantity in any equation is fixed by the relations that it bears to the known quantities, and this value can usually be found.

4. To solve an equation is to find the value of the unknown quantity This is done by a series of **transformations** by which the first member becomes the unknown quantity, and the second member becomes a known quantity, which is, therefore, the value of the unknown quantity.

TRANSFORMATIONS

5. In transforming an equation, the equality of its members must be preserved; otherwise the existing relations between the known and unknown quantities will be destroyed. Transformations are based upon the following principles:

6. In any equation:

I. The same quantity may be added to both members. For example, if 2 be added to both members of $x^2 = 16$, the members of the resulting equation, $x^2 + 2 = 18$, will be equal.

II. The same quantity may be subtracted from both members Thus, if $x^2 = 16$, then $x^2 - 2 = 14$.

III. Both members may be multiplied or both divided by the same quantity. Thus, if $x^2 = 16$, then $2x^2 = 32$ and $\frac{x^2}{2} = 8$.

IV. Both members may be raised to the same power. Thus, if $x^2 = 16$, then $x^4 = 256$.

V. Like roots of both members may be extracted. Thus, if $x^2 = 16$, then $x = 4$.

A little thought will show that none of these operations will destroy the equality of the members. In the equation $16 = 16$, for example, by I, $16 + 2 = 16 + 2$; by II, $16 - 2 = 16 - 2$; by III, $16 \times 2 = 16 \times 2$, etc. It is to be observed, however, that after any transformation, the *members* do not equal their original values. In transforming an

equation, it is not permissible to multiply or divide the given equation by 0, since principle III cannot be applied when the multiplier or divisor is 0.

7. Transposition.—In transforming an equation, it is frequently necessary to transpose, or change, a term from one member to the other. For example, in the equation $3x + 5 = 12$, let it be required to transpose the $+5$ to the second member. This may be done by *subtracting* $+5$ from both members, which, by Art. **6**, II, will not destroy the equality; thus,

$$3x + 5 = 12$$

Subtracting $+5$ from both members, $\quad 5 \quad 5$

$$3x = 12 - 5 = 7$$

Again, let it be required to transpose the -5 in $3x - 5 = 12$ to the second member. This may be done by *adding* $+5$ to both members, which, by Art. **6**, I, will not destroy the equality; thus,

$$3x - 5 = 12$$

Adding $+5$ to both members, $\quad 5 \quad 5$

$$3x = 12 + 5 = 17$$

Now, what was really accomplished in each case was to transpose 5 from the first to the second member, *with its sign changed;* and in changing a term from the second to the first member, the same operation would be performed. Hence,

8. *Any term may be transposed from either member of an equation to the other, if its sign be changed.*

9. Cancelation.—*When the same term appears with the same sign in both members of an equation, it may be canceled from both.* For, in the equation $x + a = 6 + a$, we have, by transposing the a in the first member, to the second member, $x = 6 + a - a$; whence, the a's cancel, leaving $x = 6$. It must be observed that terms will not cancel

from both members unless they have the *same* sign. Thus, in $x - a = 6 + a$, we have, by transposing the $-a$, $x = 6 + 2a$.

10. Changing Signs.—It is sometimes desirable to change the sign of a quantity in an equation from $-$ to $+$ or from $+$ to $-$. To change it, we use the following principle: *the signs of all the terms of both members of an equation may be changed without destroying the equality.* For, in the equation $-x + 4 = 10 - a$, both members may be multiplied by -1 (Art. **6**, III), giving $x - 4 = -10 + a$, or $a - 10$.

11. Clearing of Fractions.—When an equation contains fractions it must be cleared of them in order to find the value of the unknown quantity.

EXAMPLE.—Clear the equation $x + \frac{x}{2} + \frac{3x}{4} + \frac{2x}{6} = 100$ of fractions

SOLUTION—The least common denominator of the fractions is 12. By Art **6**, III, both members may be multiplied by the same quantity. Hence, multiplying each term by 12, we have $12x + \frac{12x}{2} + \frac{36x}{4} + \frac{24x}{6} = 1{,}200$. Now, reducing each fraction to its simplest form, which will not alter its value, and so will not destroy the equality of the members, we have $12x + 6x + 9x + 4x = 1{,}200$, the denominators of all the fractions having canceled.

12. *Hence, to clear an equation of fractions, multiply each term of the equation by the least common denominator.*

13. Instead of multiplying the numerators by the least common denominator and then reducing the fractions to their simplest forms, it is easier to divide the least common denominator by each denominator, and then multiply the corresponding numerators by the quotients.

EXAMPLE.—Clear the equation $\frac{2x}{x+2} = \frac{1}{2} - \frac{3x+2}{x^2-4}$ of fractions.

SOLUTION—The least common denominator is $2(x^2 - 4)$. Dividing this by $x + 2$ and multiplying $2x$ by the quotient, $2(x - 2)$, gives

$4x(x-2)$, or $4x^2-8x$, dividing $2(x^2-4)$ by 2 and multiplying 1 by the quotient, x^2-4, gives x^2-4; and dividing $2(x^2-4)$ by x^2-4 and multiplying $-(3x+2)$ by the quotient, 2, gives $-6x-4$. Hence, the equation becomes $4x^2-8x = x^2-4-6x-4$, all the denominators having canceled in the process

14. *Where a fraction is preceded by a minus sign, care must be taken to change the sign of every term of the numerator when clearing of fractions.*

EXAMPLES FOR PRACTICE

Clear the following equations of fractions

1. $x+\frac{3x}{4}+\frac{5}{7} = 16-\frac{2}{x}.$ Ans. $28x^2+21x^2+20x = 448x-56$

2. $\frac{x}{4}-\frac{x-3}{2} = \frac{a}{6}$ Ans. $3x-6x+18 = 2a$

3. $\frac{x}{a-b}-x = \frac{a-b}{a+b}-1$
Ans. $ax+bx-a^2x+b^2x = a^2-2ab+b^2-a^2+b^2$

4. $\frac{1}{(a-b)} = \frac{x}{a-b}-\frac{a+b}{x}.$ Ans. $x = x^2-a^2+b^2$

SOLUTION OF SIMPLE EQUATIONS

15. A **simple equation** is one containing only the first power of the unknown quantity, when cleared of radical and aggregation signs and fractions. It is also called an equation of the **first degree.**

16. The unknown quantity in a simple equation containing but one unknown quantity is usually represented by the letter x. Known quantities are represented by figures and by the *first* letters of the alphabet. Equations containing known quantities represented by letters are called **literal equations,** and if any literal equation be solved (Art. 4), the value of the unknown quantity will usually contain one or more of the first letters of the alphabet.

17. To solve a simple equation:

Rule.—*Clear the equation of fractions, if it has any.*

Transpose the terms containing unknown quantities to the first member, and the known terms to the second member.

Combine the terms containing the unknown quantity into one term and reduce the second member to its simplest form.

Divide both members of the resulting equation by the coefficient of the unknown quantity (Art. **6,** *III), and the second member of this last equation will be the value of the unknown quantity.*

This rule does not hold absolutely in all cases, since special methods are often used, of which the student can learn only by practice.

18. To verify the result, substitute the value of the unknown quantity in the original equation, which should then reduce so that both members will be alike. When this occurs the equation is said to be **satisfied.**

19. A **root** of an equation is the number or quantity which, when substituted for the unknown quantity, satisfies the equation. After an equation has been solved the root so obtained should always be substituted for the unknown quantity to see if it satisfies the equation; if it does the root found is correct; otherwise the work must be repeated to find the error.

20. In the following examples, the value of the unknown quantity x is to be determined. The transformations used all depend on principles explained in Arts. **5–14.**

EXAMPLE 1.—Solve the equation $20 + 5x - 3x - 18 = 10$.

SOLUTION.—Transposing 20 and -18 to the second member,

$$5x - 3x = 10 - 20 + 18$$

Combining like terms, $2x = 8$

Dividing both members by 2 (Art. **6,** III),

$$x = 4 \quad \text{Ans.}$$

To verify the result, substitute 4 for x in the original equation. (Art. **18.**) Thus,

$$20 + 5 \times 4 - 3 \times 4 - 18 = 10$$

or,

$$20 + 20 - 12 - 18 = 10$$

Combining, $10 = 10$, which proves the result

EXAMPLE 2.—Solve the equation $5x - (10 - x) = 5x + 4(x - 1)$.

SOLUTION.—Removing the parentheses,

$$5x - 10 + x = 5x + 4x - 4$$

or

$$6x - 10 = 9x - 4$$

Transposing -10 to the second member and $9x$ to the first member,

$$6x - 9x = 10 - 4$$

Combining like terms, $-3x = 6$

Changing signs to make the term containing x positive,

$$3x = -6 \quad \text{(Art. 10.)}$$

Dividing both members by 3, $x = -2$. Ans.

PROOF.— $5 \times -2 - (10 + 2) = 5 \times -2 + 4(-2 - 1)$

or

$$-10 - 10 - 2 = -10 - 8 - 4$$

Combining, $-22 = -22$, which proves the result

EXAMPLE 3.—Solve the equation

$$16 - x - \{7x - [8x - (9x - 3x - 6x)]\} = 0$$

SOLUTION.—Removing the aggregation signs,

$$16 - x - 7x + 8x - 9x + 3x - 6x = 0$$

or

$$-12x + 16 = 0$$

Transposing 16 to the second member,

$$-12x = -16$$

Dividing both members by -12,

$$x = \tfrac{16}{12} = 1\tfrac{1}{3} \quad \text{Ans.}$$

EXAMPLE 4.—Solve the equation

$$\frac{2x + 2}{2} + \frac{1}{4} = \frac{8 - 6x}{5} + \frac{2(6x + 7)}{8}$$

SOLUTION.—Reducing the first term of the first member and the last term of the second member to a simpler form, the equation becomes

$$x + 1 + \frac{1}{4} = \frac{8 - 6x}{5} + \frac{6x + 7}{4}$$

Clearing of fractions by multiplying each term of both members by 20, the least common denominator, we have

$$20x + 20 + 5 = 32 - 24x + 30x + 35$$

or

$$20x + 25 = 6x + 67$$

Transposing and uniting terms,

$$14x = 42$$

Dividing by 14, $x = 3$ Ans.

EXAMPLE 5.—Solve the equation $x + \frac{x+4}{2} - \frac{3x-4}{5} - \frac{x}{8} = 9.$

SOLUTION.—Clearing of fractions by multiplying each term by 40, the least common denominator, and remembering that the sign of the second fraction is minus (Art. **14**),

$$40x + 20x + 80 - (24x - 32) - 5x = 360$$

Removing parenthesis, transposing and uniting terms,

$$31x = 248$$

Dividing by 31, $x = 8$ Ans.

EXAMPLE 6.—Solve the equation $\frac{3}{1-x} - \frac{2}{1+x} + \frac{1}{1-x^2} = 0.$

SOLUTION.—Clearing of fractions by multiplying by $1 - x^2$, the least common denominator,

$$3(1 + x) - 2(1 - x) + 1 = 0$$
$$3 + 3x - 2 + 2x + 1 = 0$$

Uniting and transposing terms,

$$5x = -2$$
$$x = -.4 \quad \text{Ans.}$$

NOTE.— 0 multiplied or divided by any number $= 0$.

21. If the denominators in a fractional equation are partly monomial and partly polynomial, it will be easier to clear of fractions at first partially by multiplying by the least common denominator of the *monomial* denominators.

EXAMPLE 1.—Solve the equation $\frac{8x+5}{14} = \frac{4x+6}{7} - \frac{7x-3}{6x+2}.$

SOLUTION.—Clearing of fractions partially, by multiplying each term by 14, and noticing that 2 may be canceled from the denominator of the second fraction of the second member when multiplying by 14,

$$8x + 5 = 8x + 12 - \frac{49x - 21}{3x + 1}$$

Transposing and uniting the terms (Art. **9**),

$$\frac{49x - 21}{3x + 1} = 7$$

Clearing of fractions by multiplying each term by $3x+1$,

$$\begin{aligned} 49x - 21 &= 21x + 7 \\ 28x &= 28 \\ x &= 1 \quad \text{Ans.} \end{aligned}$$

EXAMPLE 2.—Solve the equation $1 + \dfrac{3}{x-1} = \dfrac{3 + \dfrac{4-x}{1-x}}{3}$.

SOLUTION.—Simplifying the second member by multiplying both numerator and denominator of the fraction by $1 - x$,

$$1 + \frac{3}{x-1} = \frac{3(1-x) + 4 - x}{3(1-x)}$$

Changing the signs of the first fraction so as to make the denominator $1 - x$, and clearing of fractions by multiplying by $3(1-x)$,

$$3(1-x) - 9 = 3(1-x) + 4 - x$$

Canceling $3(1-x)$ from both members and transposing,

$$x = 13 \quad \text{Ans.}$$

22. When powers of the unknown quantity higher than the first appear in an equation, they will often cancel, the equation thus reducing to a simple one.

EXAMPLE.—Solve the equation

$$(x+3)^2 - 3x(4x+1) = 5x^2 - (4x-5)^2$$

SOLUTION.—Performing the operations indicated,

$$x^2 + 6x + 9 - 12x^2 - 3x = 5x^2 - (16x^2 - 40x + 25)$$

Removing the parenthesis and transposing terms,

$$x^2 + 6x - 12x^2 - 3x - 5x^2 + 16x^2 - 40x = -25 - 9$$

Combining like terms, $\quad -37x = -34$

Dividing by -37, $\quad x = \frac{34}{37}$ Ans.

23. In **literal equations** (Art. **16**), the terms containing known or unknown quantities cannot always be combined into one. In solving, all terms containing unknown quantities must be brought into the first member without regard to whether they contain known quantities.

EXAMPLE 1.—Solve the literal equation $2ax - 3b = x + c - 3ax$.

SOLUTION.—Transposing the terms containing the unknown quantities to the first member and the remaining terms to the second member, and combining like terms,

$$5ax - x = 3b + c$$

Factoring $5ax - x$ with a view to bringing x alone in the first member,

$$(5a-1)x = 3b + c$$

The coefficient of x is now $5a-1$, this being considered as one quantity.

Dividing by $5a-1$, $\qquad x = \dfrac{3b+c}{5a-1}$ Ans.

PROOF.—Since the original equation is equivalent to $5ax - x = 3b + c$, it will be sufficient to satisfy this equation. Hence, substituting the value of x,

$$\frac{5a(3b+c)}{5a-1} - \frac{3b+c}{5a-1} = 3b + c$$

or

$$\frac{(5a-1)(3b+c)}{5a-1} = 3b + c$$

Canceling the $5a-1$,

$$3b + c = 3b + c$$

EXAMPLE 2.—Solve the equation

$$(x+a)(x-b) - (x-a)(x+b) = a^2 - b^2$$

SOLUTION.—Performing the operations indicated,

$$x^2 + ax - bx - ab - (x^2 - ax + bx - ab) = a^2 - b^2$$

Combining like terms, $2ax - 2bx = a^2 - b^2$

whence, $\qquad 2(a-b)x = a^2 - b^2$

or $$x = \frac{a^2-b^2}{2(a-b)} = \frac{(a+b)(a-b)}{2(a-b)} = \frac{(a+b)}{2} \text{ Ans.}$$

EXAMPLE 3.—Solve the equation, $\dfrac{3x+1}{x+1} = \dfrac{3bx-2a+c}{b(x+1)-a}$

SOLUTION.—Clearing of fractions,

$$(3x+1)[b(x+1)-a] = (x+1)(3bx-2a+c)$$

or

$$3bx(x+1) - 3ax + b(x+1) - a = 3bx(x+1) - (2a-c)(x+1)$$

Canceling $3bx(x+1)$ from both members,

$$-3ax + bx + b - a = -2ax + cx - 2a + c$$

Transposing and uniting terms,

$$-ax + bx - cx = -a - b + c$$

Changing signs and factoring,

$$(a-b+c)x = a + b - c$$

whence, $$x = \frac{a+b-c}{a-b+c} \text{ Ans.}$$

EXAMPLES FOR PRACTICE

Solve the following equations:

1. $16 - 3x = 13 - 6x.$ Ans. $x = -1$
2. $3(4x - 5) + 6 = 1 + 2x.$ Ans. $x = 1$
3. $6(5 - 2x) = 6 - 2(x - 2).$ Ans. $x = 2$
4. $\frac{2x}{3} - \frac{4x}{3} = 5 - \frac{3x}{4}.$ Ans. $x = 60$
5. $\frac{x+1}{3} - \frac{x+4}{5} = 16 - \frac{x+3}{4}.$ Ans. $x = 41$
6. $\frac{x}{3} - \frac{x^2 - 5x}{3x - 7} = \frac{2}{3}.$ Ans. $x = -7$
7. $\frac{5 - 2x}{x + 1} - \frac{3 - 2x}{x + 4} = 0.$ Ans. $x = 4\frac{1}{4}$
8. $2x - 4a = 3ax + a^2 - a^2x.$ Ans. $x = \frac{a^2 + 4a}{a^2 - 3a + 2}$
9. $\frac{ax + 2x}{5a} - \frac{a^2 + 4a + 4}{4b} = 0.$ Ans. $x = \frac{5a^2 + 10a}{4b}$

SUGGESTION.—Transposing the second term to the second member,

$$\frac{ax + 2x}{5a} = \frac{a^2 + 4a + 4}{4b} = \frac{(a + 2)^2}{4b}$$

Multiplying both sides by $5a$,

$$ax + 2x = \frac{5a(a + 2)^2}{4b}$$

Solving for x,

$$x = \frac{5a(a + 2)^2}{(a + 2)4b} = \frac{5a(a + 2)}{4b} = \frac{5a^2 + 10a}{4b}$$

10. $\frac{a(c^2 + x^2)}{cx} = ab + \frac{ax}{c}.$ Ans. $x = \frac{c}{b}$

PROBLEMS LEADING TO SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY

24. There are two steps in the solution of problems by algebra:

First.—The relations which exist between the known and the unknown quantities, that is, between those whose values are given in the problem and those whose values are required, must be stated by one or more equations. This is called the **statement** of the problem.

Second.—The resulting equation or equations must be solved, giving the values of the required quantities.

25. The ability to state a problem by means of an equation depends upon the ingenuity of the operator and his ability to reason, rather than upon his knowledge of algebra. No definite rule can be given for making the statement, but in general, where there is only one unknown quantity in a problem:

Decide what quantity it is whose value is to be found. This will be the unknown quantity, or the answer. Then represent the unknown quantity by x and form an equation that will indicate the relations between the known and the unknown quantities as stated in the problem.

It will thus be seen that by the algebraic method, the answer to a problem is used in the solution and operated upon as though it were a known quantity, which is one great advantage over the arithmetical method.

NOTE.—The equation will also indicate the operations that would be performed in proving the statement made in the problem, were the answer known. Hence, the equation may often be formed by noticing what operations would be performed upon the answer in proving.

EXAMPLE 1.—Find such a number that, when 14 is added to its double, the sum shall be 30.

SOLUTION.—The quantity whose value is required is the number itself. As this is the unknown quantity, let $x =$ the number, whence $2x$ must be double the number. Now the problem states that when 14 is added to double the number the sum will be 30. In other words, when 14 is added to $2x$, the sum will be 30. Hence, the statement of the problem in the form of an equation is

$$2x + 14 = 30$$

Whence, solving, $x = 8$ Ans.

EXAMPLE 2.—Find a number which, when multiplied by 4, will exceed 40 as much as it is now below 40.

SOLUTION.—Let $x =$ the required number, which, when multiplied by 4, becomes $4x$. According to the conditions of the problem, the amount by which 4 times the required number, or $4x$, exceeds 40 is equal to the amount that the number itself, or x, is below 40.

But $4x - 40$ is the amount by which $4x$ exceeds 40, and $40 - x$ is the amount by which x is below 40.

Hence, by the conditions, we have the statement,

$$4x - 40 = 40 - x$$

Transposing and uniting, $5x = 80$

or $x = 16$ Ans.

EXAMPLE 3.—Two loads of brick together weigh 4,000 lb.; but if 500 lb. be transferred from the smaller to the larger load, the latter will weigh 7 times as much as the former. How much does each load weigh?

SOLUTION.—If the weights of the two loads were known and it was desired to prove the correctness of the example, we should add 500 lb. to the weight of the larger load and subtract 500 lb. from the weight of the smaller load, as stated in the example. The larger load should then weigh 7 times as much as the smaller. To obtain the equation, the same thing is done by letting $x =$ the weight of one load, whence $4,000 - x$ equals the weight of the other load.

Let $x =$ the weight of the smaller load.

Then, $4,000 - x =$ the weight of the larger load.

Also, $x - 500 =$ the weight of the smaller load after transferring 500 lb.

And $4,000 - x + 500 =$ the weight of the larger load after transferring 500 lb.

By the conditions, the larger load now weighs 7 times as much as the smaller.

Hence, $7(x - 500) = 4,000 - x + 500$

Solving, $7x - 3,500 = 4,500 - x$

or $8x = 8,000$

whence, $x = 1,000$ lb. = weight of smaller load

and $4,000 - x = 3,000$ lb. = weight of larger load } Ans.

PROOF.— $1,000 - 500 = 500 =$ weight of the smaller load, and $3,000 + 500 = 3,500 =$ weight of the larger load after the 500 pounds have been transferred; $3,500 \div 500 = 7$.

EXAMPLE 4.—The circumference of the fore wheel of a carriage is 10 feet, and of the hind wheel 12 feet. What distance has the carriage traveled, when the fore wheel has made 8 more turns than the hind wheel?

SOLUTION.—In this example the distance traveled is not known, but is required to be found. Suppose that the distance is known, and that it equals x feet, and that we wish to see whether the statement is true that the fore wheel makes 8 more revolutions than the hind wheel in

passing over x feet. The number of revolutions of the fore wheel is evidently $\frac{x}{10}$, and of the hind wheel, $\frac{x}{12}$. The example states that the difference between them equals 8.

Hence, $$\frac{x}{10} - \frac{x}{12} = 8 \qquad (1)$$

Multiply both members of the equation by 60.

$$6x - 5x = 480$$
$$x = 480 \text{ ft.} \quad \text{Ans.}$$

PROOF.— $$\frac{480}{10} = 48 = \text{revolutions of fore wheel}$$
$$\frac{480}{12} = 40 = \text{revolutions of hind wheel}$$
$$48 - 40 = 8. \quad \text{Compare this proof with (1)}$$

EXAMPLE 5.—A water cistern connected with three pipes can be filled by one of them in 80 minutes, by another in 200 minutes, and by the third in 300 minutes. In what time will the cistern be filled when all three pipes are open at once?

SOLUTION.—Here the unknown quantity is the number of minutes required to fill the cistern by all three pipes together. Supposing this to be x minutes, the example may be proved by noticing that the sum of the quantities of water flowing through each pipe separately in a given length of time, as 1 minute, must be equal to the quantity flowing through all three together in the same length of time. According to the problem, the quantity discharged by the first pipe in one minute would be $\frac{1}{80}$, by the second $\frac{1}{200}$, and by the third $\frac{1}{300}$ of the contents of the cistern. In like manner the quantity discharged by all three at once in one minute would be $\frac{1}{x}$. Then, if the example is stated correctly,

$$\frac{1}{80} + \frac{1}{200} + \frac{1}{300} = \frac{1}{x}$$

Clearing of fractions,

$$x(30 + 12 + 8) = 2{,}400$$

or $$50x = 2{,}400$$

whence, $$x = 48 \text{ minutes} \quad \text{Ans.}$$

EXAMPLE 6.—A man rows a boat a certain distance *with* the tide, at the rate of $6\frac{2}{3}$ miles an hour, and returns at the rate of $3\frac{1}{3}$ miles an hour, *against* a tide half as strong. If the man is pulling at a uniform rate, what is the velocity of the stronger tide?

SOLUTION.—If the following statement is not clear, the student should reason it out for himself in a manner similar to that used in the last three examples.

Let x = number of miles per hour that the stronger tide is running, then $\frac{x}{2}$ = number of miles per hour that the weaker tide is running.

Hence, $6\frac{2}{3} - x$ and $3\frac{1}{3} + \frac{x}{2}$ are expressions for the rate at which the man is pulling. But, as he is pulling at a constant rate all the time, these expressions must be equal. Hence,

$$6\tfrac{2}{3} - x = 3\tfrac{1}{3} + \frac{x}{2}$$

or
$$\frac{20}{3} - x = \frac{10}{3} + \frac{x}{2}$$

Clearing of fractions, $40 - 6x = 20 + 3x$

or $-9x = -20$

whence, $x = 2\frac{2}{9}$ miles per hour Ans.

EXAMPLES FOR PRACTICE

Solve the following examples:

1. The greater of two numbers is four times the lesser number, and their sum is 400; what are the numbers? Ans. 80 and 320

2. A farmer has 108 animals, consisting of horses, sheep, and cows. He has four times as many cows as horses, lacking 8, and five times as many sheep as horses, lacking 4; how many has he of each kind?

Ans. $\begin{cases} 12 \text{ horses} \\ 40 \text{ cows} \\ 56 \text{ sheep} \end{cases}$

3. A can do a piece of work in 8 days, and B can do it in 10 days; in what time can they do it working together? Ans. $4\frac{4}{9}$ days

4. Find five consecutive numbers whose sum is 150.

Ans. $28 + 29 + 30 + 31 + 32$

5. A boat whose rate of sailing is 6 miles per hour in still water moves down a stream which flows at the rate of 3 miles per hour, and returns, making the round trip in 8 hours; how far did it go down the stream? Ans. 18 mi.

QUADRATIC EQUATIONS

26. A **quadratic equation** is one in which the *square* is the highest power of the unknown quantity when simplified as stated in Art. **15.** It is also called an equation of the **second degree.**

27. A **pure quadratic equation** is one which contains the square only of the unknown quantity, as $x^2 + 2ab = 10$.

28. An **affected quadratic equation** is one containing both the square and the first power of the unknown quantity, as $x^2 + 2x = 6$.

29. By the processes used to reduce simple equations, any pure quadratic equation may be reduced to an equation having the square of the unknown quantity alone in the first member, and some known quantity in the second member, as in $x^2 = a$, where x^2 is the square of the unknown quantity and a is a known quantity. The value of the unknown quantity may then be found by extracting the square root of both members, which, by Art. **6,** V, will not destroy the equality of the equation. By referring to Case (7), Art. **64,** Part 2, it will be seen that after extracting the square root, each member should be written with the $\pm$ sign. Thus, extracting the square root of both members of $x^2 = a$, we have $\pm x = \pm\sqrt{a}$. This may be taken in four ways, namely, that

$$\begin{aligned} +x &= +\sqrt{a} \\ +x &= -\sqrt{a} \\ -x &= -\sqrt{a} \\ -x &= +\sqrt{a} \end{aligned}$$

But by Art. **10,** the signs of both members of the last two equations may be changed, making $+x = +\sqrt{a}$ and

$+x = -\sqrt{a}$, the same as in the first two equations. Hence, the equation $x^2 = a$ has the two values,

$$x = +\sqrt{a}$$

and $$x = -\sqrt{a}$$

and these may be expressed by writing x in the first member without any sign (plus understood), and writing the square root of a in the second member with the $\pm$ sign, thus,

$$x = \pm\sqrt{a}$$

30. From the foregoing, we have the following rule for solving a pure quadratic equation:

Rule.—*Reduce the given equation to the form of $x^2 = a$ (Art.* **29**), *and extract the square root of both members, writing the $\pm$ sign before the square root of the second member.*

NOTE.—The **root** of an equation is the value of the unknown quantity. From this it will be seen that a simple equation has one root, and a quadratic equation has two roots. In general, any equation has as many roots as there are units in the exponent of the unknown quantity.

EXAMPLE 1.—Solve the equation $\frac{x^2}{16} - \frac{x^2 - 3}{5} = \frac{1}{20}$.

SOLUTION.—Clearing of fractions by multiplying each term by 80,

$$5x^2 - 16(x^2 - 3) = 4$$

Transposing and uniting, $-11x^2 = -44$,

or $$x^2 = 4$$

Extracting the square root of both members,

$$x = \pm 2 \quad \text{Ans.}$$

EXAMPLE 2.—Solve the equation

$$\frac{\sqrt{x-2}}{\sqrt{x+2}} + \frac{\sqrt{x+2}}{\sqrt{x-2}} = 4$$

SOLUTION.—Clearing of fractions by multiplying each term by $\sqrt{x+2} \times \sqrt{x-2}$,

$$x - 2 + x + 2 = 4\sqrt{x+2} \times \sqrt{x-2}$$

or, $$2x = 4\sqrt{x^2 - 4}$$

Dividing by 2, $$x = 2\sqrt{x^2 - 4}$$

Squaring, $$x^2 = 4(x^2 - 4)$$

or $$x^2 = 4x^2 - 16$$

whence, $$-3x^2 = -16$$

and $$x^2 = \frac{16}{3}$$

Extracting the square root of both members,

$$x = \pm 4\sqrt{\tfrac{1}{3}} \quad \text{Ans.}$$

NOTE.—That $\sqrt{x+2} \times \sqrt{x-2} = \sqrt{x^2-4}$ is readily seen from the following: Using fractional exponents $\sqrt{x-2} \times \sqrt{x+2} = (x-2)^{\frac{1}{2}}(x+2)^{\frac{1}{2}}$. Since $25^{\frac{1}{2}} \times 36^{\frac{1}{2}} = 5 \times 6 = 30$ and since $(25 \times 36)^{\frac{1}{2}} = 900^{\frac{1}{2}} = 30$, it follows that $25^{\frac{1}{2}} \times 36^{\frac{1}{2}} = (25 \times 36)^{\frac{1}{2}}$; and since any numbers whatever may be substituted for 25 and 36, a and b may be substituted also, and $a^{\frac{1}{2}} \times b^{\frac{1}{2}} = (ab)^{\frac{1}{2}}$. Now letting $a = x-2$ and $b = x+2$, $a^{\frac{1}{2}} \times b^{\frac{1}{2}} = (x-2)^{\frac{1}{2}}(x+2)^{\frac{1}{2}} = [(x-2)(x+2)]^{\frac{1}{2}} = (x^2-4)^{\frac{1}{2}} = \sqrt{x^2-4}$.

EXAMPLES FOR PRACTICE

Solve the following equations:

1. $3x^2 - 57 - 4x^2 = -8x^2 + 6$. Ans. $x = \pm 3$
2. $\dfrac{1}{2x^2} + 7 = \dfrac{9}{4x^2}$. Ans. $x = \pm \frac{1}{2}$
3. $35 - \dfrac{x^2+50}{5} = x^2 - \dfrac{x^2-10}{3}$. Ans. $x = \pm 5$
4. $x\sqrt{6} + x^2 = 1 + x^2$. Ans. $x = \pm \frac{1}{2}$

AFFECTED QUADRATIC EQUATIONS

31. Every affected quadratic equation may be reduced to the form

$$x^2 \pm px = \pm q$$

in which the term containing x^2 is positive and the coefficient is 1; the term containing x is positive or negative and the coefficient has any value; and the remaining term q has any value and is positive or negative. For example, suppose it is required to bring the equation $ax^2 - bx + cx - x^2 + 3 = d$ into the required form. First collect the terms containing x^2 and factor; then the terms containing x and factor; then the terms that do not contain x^2 or x after transposing them to the second member. Lastly divide by the coefficient of x^2. Thus,

$$ax^2 - x^2 + cx - bx = d - 3$$

$$(a-1)x^2 + (c-b)x = d - 3$$

$$x^2 + \frac{c-b}{a-1}x = \frac{d-3}{a-1}$$

Here x^2 is positive, and the coefficient is 1; the coefficient of x is $\frac{c-b}{a-1}$, which may be put equal to p; and the known term (usually called the **absolute term,** because it does not change for any value of x) is $\frac{d-3}{a-1}$, which may be represented by q. The equation is now of the required form.

32. Any equation of the form $x^2 \pm px = \pm q$ may be solved, that is, the values of x (the *roots* of the equation) may be found, whether the coefficients are numerical or literal, by the following formula:

$$x = \mp \tfrac{1}{2}(p \pm \sqrt{p^2 \pm 4q})^*$$

The $\mp$ sign is read *minus or plus*, and is a combination of the minus and plus signs. In this formula, *the minus sign before the parenthesis is used if the coefficient of x in the original equation is positive, and the positive sign is used if this coefficient is negative;* the plus sign between p^2 and $4q$ is used if q in the original equation is positive, and the negative sign is used if q is negative. The double sign before the radical indicates that there are two values of x, one of which is equal to one-half of p plus the radical, and the other to one-half of p minus the radical.

EXAMPLE.—Solve the equation $4x^2 - 16x - 128 = 0$.

SOLUTION.— $4x^2 - 16x - 128 = 0$

Transposing 128, $4x^2 - 16x = 128$

Dividing by 4, $x^2 - 4x = 32$

The equation is now in the required form, and p in the formula equals 4, while $q = 32$. Since p is negative, use the positive sign before the parenthesis, and since q is positive, use the positive sign under the radical sign. Substituting,

$$\begin{aligned} x &= \tfrac{1}{2}(4 \pm \sqrt{4^2 + 4 \times 32}) \\ &= \tfrac{1}{2}(4 \pm \sqrt{16 + 128}) \\ &= \tfrac{1}{2}(4 \pm \sqrt{144}) \\ &= \tfrac{1}{2}(4 \pm 12) \\ &= \tfrac{1}{2}(4 + 12), \text{ or } \tfrac{1}{2}(4 - 12) \\ &= 8 \text{ or } -4 \quad \text{Ans.} \end{aligned}$$

* For proof see page 40.

33. The result just obtained may be proved in two ways: *First*, by substituting the values found for x in the original equation; if both satisfy the equation, the results are correct. *Second*, put the original equation in the form $x^2 \pm px \pm q = 0$, by transposing the absolute term; then form two binomial factors by adding to x the roots *with their signs changed;* the product of these factors must equal the first member of the equation, if the work is correct.

Applying the first proof to the last example,

$$4(+8)^2 - 16(+8) - 128 = 256 - 128 - 128 = 0$$
$$4(-4)^2 - 16(-4) - 128 = 64 + 64 - 128 = 0$$

Applying the second proof to the same example, the roots with their signs are -8 and 4; adding these to x, the sums are $x-8$ and $x+4$. Treating these binomials as factors and expanding $(x-8)(x+4) = x^2 - 4x - 32$, which is the value of the first member of the equation $4x^2 - 16x - 128 = 0$ when reduced to the form $x^2 \pm px \pm q = 0$, by dividing both members by 4. It is to be noted that 0 multiplied or divided by any finite quantity is zero.

34. If any equation of the form $x^2 \pm px \pm q = 0$ can be factored (and every such equation can be factored), either factor can be placed equal to zero, and by transposing the absolute term the value of x can be found. For example, in the last article $x^2 - 4x - 32 = 0$; hence $(x-8)(x+4) = 0$, from which $x - 8 = \frac{0}{x+4} = 0$, and $x = 8$, or $x + 4 = \frac{0}{x-8} = 0$, and $x = -4$.

This fact gives an easy method of determining the roots by inspection when the equation has numerical coefficients and the roots are integral or fractional. It is evident, as will be seen by actual multiplication, that the product of the absolute terms of the factors must equal the absolute term of the given equation; also that the sum of the absolute terms of the factors must be equal to

the coefficient of x; in both cases the sign of the term is supposed to be included in the statement. Consider now the equation

$$x^2 - 4x - 32 = 0$$

The absolute term -32 is obtained by multiplying two numbers with unlike signs; the coefficient of x, which is -4, is obtained by adding either two negative quantities or a positive and a negative quantity, the negative quantity being the greater to obtain the minus sign. The following are all the pairs of integral factors of -32, whose product will equal -32, together with their sums:

Product	*Sum*	*Product*	*Sum*
$-1 \times 32 = -32$	$-1 + 32 = 31$	$1 \times -32 = -32$	$1 + (-32) = -31$
$-2 \times 16 = -32$	$-2 + 16 = 14$	$2 \times -16 = -32$	$2 + (-16) = -14$
$-4 \times 8 = -32$	$-4 + 8 = 4$	$4 \times -8 = -32$	$4 + (-8) = -4$

In the last case, both conditions are fulfilled; hence, $x^2 - 4x - 32 = (x - 8)(x + 4) = 0$, from which $x = 8$ or -4. It is well in all cases to attempt the solution by inspection before applying the formula, since if a solution is possible by this method the work is greatly reduced. This method also proves the work by simply multiplying the factors.

EXAMPLE.—Solve by inspection $13x - x^2 = -14$.

SOLUTION.—Bring the equation into the required form by changing all the signs and transposing the absolute term.

$$x^2 - 13x - 14 = 0$$

The only pairs of integral factors of 14 are 1×14 and 2×7. The coefficient of x is -13, and since $-14 + 1 = -13$, the factors are evidently -14×1. Hence,

$$(x - 14)(x + 1) = 0, \text{ and } x = 14 \text{ or } -1 \quad \text{Ans.}$$

REMARK.—If the formula had been used in solving the last example the work would have been as follows:

$$x^2 - 13x = 14$$

$$\begin{aligned} x &= \tfrac{1}{2}(13 \pm \sqrt{13^2 + 4 \times 14}) \\ &= \tfrac{1}{2}(13 \pm \sqrt{225}) \\ &= \tfrac{1}{2}(13 \pm 15) \\ &= 14 \text{ or } -1 \end{aligned}$$

35. The principles given in Art. **34** may be obtained directly from the following data for writing out the product of any two binomials, the first terms of which are alike:

Add the square of the first term, the product of the first term and the sum of the second terms, and the product of the second terms.

EXAMPLE.—Write out the products of $x-3$ and $x-6$; also, of x^2-a and x^2+b.

SOLUTION.—For the first case, the square of the first term is x^2; the sum of the second terms is $-3-6=-9$; and the product of the second terms is $-3\times-6=18$; hence, $(x-3)(x-6)=x^2-9x+18$. Ans.

For the second case, the square of the first term is x^4; the sum of the second term is $-a+b=-(a-b)$ or $(b-a)$; the product of the second terms is $-ab$; hence,

$$(x^2-a)(x^2+b)=x^4-(a-b)x^2-ab \text{ or } x^4+(b-a)x^2-ab \quad \text{Ans.}$$

36. Several examples will now be given showing the application of the foregoing methods to the solution of typical examples.

EXAMPLE 1.—Solve the equation $-3x^2-7x=\frac{10}{3}$.

SOLUTION.—Dividing both members by -3 to make x^2 stand alone and positive, $x^2+\frac{7}{3}x=-\frac{10}{9}$.

From the formula,

$$\begin{aligned} x &= -\tfrac{1}{2}\left(\tfrac{7}{3}\pm\sqrt{\tfrac{49}{9}-\tfrac{40}{9}}\right) \\ &= -\tfrac{1}{2}\left(\tfrac{7}{3}\pm 1\right) \\ &= -\tfrac{5}{3} \text{ or } -\tfrac{2}{3} \quad \text{Ans.} \end{aligned}$$

The example may also be solved by inspection, as follows:

$$x^2+\tfrac{7}{3}x+\tfrac{10}{9}=0$$

The absolute term $\frac{10}{9}$ is equal to $\frac{2\times5}{3\times3}=\frac{2}{3}\times\frac{5}{3}$, and $\frac{2}{3}+\frac{5}{3}=\frac{7}{3}$; hence, $(x+\frac{2}{3})(x+\frac{5}{3})=0$, and $x=-\frac{2}{3}$ or $-\frac{5}{3}$. Ans.

EXAMPLE 2.—Solve the equation $x-\dfrac{x^3-8}{x^2+5}=2$.

SOLUTION.—Clearing of fractions,

$$x^3+5x-x^3+8=2x^2+10$$

Transposing and uniting terms,

$$-2x^2+5x=2$$

Dividing by -2, $\qquad x^2-\frac{5}{2}x=-1$

From the formula, $x = \frac{1}{2}\left(\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4}\right)$
$= 2$ or $\frac{1}{2}$ Ans.

Solving by inspection,

$$x^2 - \tfrac{5}{2}x + 1 = 0$$

$$1 = \tfrac{2}{2} = \frac{2 \times 1}{1 \times 2} = 2 \times \tfrac{1}{2};\ -2 + (-\tfrac{1}{2}) = -2\tfrac{1}{2} = -\tfrac{5}{2}$$

hence, $(x - 2)(x - \frac{1}{2}) = 0$, and $x = 2$ or $\frac{1}{2}$ Ans.

EXAMPLE 3.—Solve the literal equation $acx^2 - bcx + adx = bd$.

SOLUTION.—Reducing the equation so that the first member will contain two terms, one with x^2 and one with x,

$$acx^2 - (bc - ad)x = bd$$

Dividing by ac, $x^2 - \dfrac{bc - ad}{ac}x = \dfrac{bd}{ac}$

From the formula,

$$x = \tfrac{1}{2}\left(\frac{bc - ad}{ac} \pm \sqrt{\left(\frac{bc - ad}{ac}\right)^2 + \frac{4bd}{ac}}\right)$$

$$= \tfrac{1}{2}\left(\frac{bc - ad}{ac} \pm \sqrt{\frac{b^2c^2 - 2bcad + a^2d^2}{a^2c^2} + \frac{4bdac}{a^2c^2}}\right)$$

$$= \tfrac{1}{2}\left(\frac{bc - ad}{ac} \pm \sqrt{\frac{b^2c^2 + 2bcad + a^2d^2}{a^2c^2}}\right)$$

$$= \tfrac{1}{2}\left(\frac{bc - ad}{ac} \pm \sqrt{\frac{(bc + ad)^2}{(ac)^2}}\right)$$

$$= \tfrac{1}{2}\left(\frac{bc - ad}{ac} \pm \frac{bc + ad}{ac}\right)$$

$$= \tfrac{1}{2}\left(\frac{2bc}{ac}\right) = \frac{b}{a} \text{ or } \tfrac{1}{2}\left(-\frac{2ad}{ac}\right) = -\frac{d}{c} \quad \text{Ans.}$$

The example may also be solved by inspection.

$$x^2 - \frac{bc - ad}{ac}x - \frac{bd}{ac} = 0$$

The coefficient of x is $-\dfrac{bc - ad}{ac}$, which is equal to $-\left(\dfrac{bc}{ac} - \dfrac{ad}{ac}\right)$ $= -\left(\dfrac{b}{a} - \dfrac{d}{c}\right) = -\dfrac{b}{a} + \dfrac{d}{c}$; the product of these two fractions is $-\dfrac{b}{a} \times \dfrac{d}{c} = -\dfrac{bd}{ac}$, which is the same as the absolute term; hence,

$$\left(x - \frac{b}{a}\right)\left(x + \frac{d}{c}\right) = 0, \text{ and } x = \frac{b}{a} \text{ or } -\frac{d}{c} \quad \text{Ans.}$$

EXAMPLE 4.—Solve for x in the equation $80 - 3x^2 - 2x = -5$.

SOLUTION.—Transposing the known term in the left-hand member, $-3x^2 - 2x = -85$. Dividing by the coefficient of x (which is -3 in this case), the equation becomes $x^2 + \frac{2}{3}x = \frac{85}{3}$.

From the formula, $x = -\frac{1}{2}\left(\frac{2}{3} \pm \sqrt{\frac{4}{9} + \frac{340}{3}}\right)$

$= -\frac{1}{2}\left(\frac{2}{3} \pm \frac{32}{3}\right)$

$= -\frac{17}{3}$ or $\frac{15}{3} = 5$ Ans.

By inspection, the factors of $\frac{85}{3}$ are $\frac{17}{3}$ and 5; but $5 = \frac{15}{3}$; and since $\frac{17}{3} + (-\frac{15}{3}) = \frac{2}{3}$, it is evident that $x^2 + \frac{2}{3}x - \frac{85}{3} = (x + \frac{17}{3})(x - 5) = 0$, and $x = -\frac{17}{3}$ or 5. Ans.

Example 5.—Find the value of x in the equation $\frac{x}{x+a} = \frac{b}{x-b}$.

Solution.—Clearing of fractions, $x(x-b) = b(x+a)$ or $x^2 - bx = bx + ab$. The term bx in the right-hand member must be transposed to the other side so that only the known term shall be on the right. The equation then becomes $x^2 - bx - bx = ab$, or

$$x^2 - 2bx = ab$$

From the formula,

$$x = \tfrac{1}{2}(2b \pm \sqrt{4b^2 + 4ab})$$
$$= \tfrac{1}{2}(2b \pm \sqrt{4(b^2 + ab)})$$
$$= \tfrac{1}{2}(2b \pm 2\sqrt{b^2 + ab})$$
$$= b \pm \sqrt{b^2 + ab} \quad \text{Ans.}$$

This example cannot be solved by inspection, since the required factors, which are $x - b + \sqrt{b^2 + ab}$ and $x - b - \sqrt{b^2 + ab}$ can be determined only by the aid of the formula. That the result is correct may be proved by multiplying the factor. Thus, $(x - b + \sqrt{b^2 + ab})(x - b - \sqrt{b^2 + ab}) = [x - (b - \sqrt{b^2 + ab})] \times [x - (b + \sqrt{b^2 + ab})] = x^2 - (b - \sqrt{b^2 + ab} + b + \sqrt{b^2 + ab})\,x + (b - \sqrt{b^2 + ab})(b + \sqrt{b^2 + ab})$ (see Art. **35**) $= x^2 - 2bx + b^2 - (b^2 + ab)$ (see Art. **27**, Part 2) $= x^2 - 2bx - ab$, which is the same as the original equation with the absolute term transposed to the first member.

Example 6.—Find the positive value of T in the equation

$$2.03222 = 6.1007 - \frac{2,719.78}{T} - \frac{400,215}{T^2}$$

Solution.—Clearing of fractions and transposing,

$$4.06848\,T^2 - 2,719.78\,T = 400,215$$

Dividing by 4.06848, $T^2 - 668.500\,T = 98,369.7$

Applying the formula,

$$T = \tfrac{1}{2}(668.5 \pm \sqrt{668.5^2 + 4 \times 98,369.7}) = 792.609 \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Solve the following equations:

1. $x^2 + 2x = 35.$ Ans. $x = 5$ or -7
2. $9x^2 + 6x = 15.$ Ans. $x = 1$ or $-1\frac{2}{3}$
3. $5x^2 - 24x = 5.$ Ans. $x = 5$ or $-\frac{1}{5}$
4. $x + \dfrac{24}{x-1} = 3x - 4.$ Ans. $x = 5$ or -2
5. $-5x^2 + 9x = 2\frac{1}{4}.$ Ans. $x = \frac{3}{2}$ or $\frac{3}{10}$
6. $\dfrac{x}{x+1} + \dfrac{x+1}{x} = \frac{13}{6}.$ Ans. $x = 2$ or -3
7. $\dfrac{9x}{12x + 6b} = \dfrac{3b}{4x - 2b}.$ Ans. $x = \dfrac{b}{4}(3 \pm \sqrt{17})$
8. $\dfrac{2x(a-x)}{3a - 2x} = \dfrac{a}{4}.$ Ans. $x = \dfrac{3a}{4}$ or $\dfrac{a}{2}$

EQUATIONS IN THE QUADRATIC FORM

37. An equation is in the *quadratic form* when it contains only two powers of the unknown quantity, *and the exponent of one power is twice as great as the exponent of the other.* Such equations are solved by the rules for quadratics.

EXAMPLE 1.—Solve the equation $x^4 + 4x^2 = 12.$

SOLUTION.—By inspection,

$$x^4 + 4x^2 - 12 = (x^2 - 2)(x^2 + 6) = 0$$

whence, $$x^2 = 2 \text{ or } -6$$

Extracting the square root,

$$x = \pm\sqrt{2} \text{ or } \pm\sqrt{-6} \quad \text{Ans.}$$

EXAMPLE 2.—Solve the equation $x^6 + 20x^3 - 10 = 59.$

SOLUTION.—Transposing the 59,

$$x^6 + 20x^3 - 69 = 0$$

By inspection, $x^6 + 20x^3 - 69 = (x^3 + 23)(x^3 - 3) = 0;$

whence, $$x^3 = 3 \text{ or } -23$$

Extracting the cube root,

$$x = \sqrt[3]{3} \text{ or } \sqrt[3]{-23} = -\sqrt[3]{23} \quad \text{Ans.}$$

EXAMPLE 3.—Solve the equation $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$.

SOLUTION.—Using the formula, because the factors are not easily found,

$$x^{\frac{3}{5}} = -\tfrac{1}{2}\left(1 \pm \sqrt{1 + 3{,}024}\right) = -\tfrac{1}{2}(1 \pm 55) = -28 \text{ or } 27$$

Now, to obtain a value for x, we must extract the cube root of both members and then raise both members to the 5th power. This will clear x of its fractional exponent.

Extracting the cube root, $x^{\frac{1}{5}} = 3$ or $-\sqrt[3]{28}$.

Raising to the fifth power, $x = 243$ or $-\sqrt[3]{28^5}$. Ans.

EXAMPLES FOR PRACTICE

Solve the following equations:

1. $x^4 + 4x^2 = 117$. Ans. $x = \pm 3$ or $\pm \sqrt{-13}$
2. $x^6 - 2x^3 = 48$. Ans. $x = 2$ or $-\sqrt[3]{6}$
3. $x^6 - 8x^3 = 513$. Ans. $x = 3$ or $-\sqrt[3]{19}$
4. $x^3 - x^{\frac{3}{2}} = 56$. Ans. $x = 4$ or $(-7)^{\frac{2}{3}}$

PROBLEMS LEADING TO QUADRATIC EQUATIONS

38. In quadratics, where two answers are obtained by solving equations, it is usually the case that only one answer, the positive value, is required. In some instances, however, the negative value is the one sought. In works treating on higher mathematics, the negative value is used as frequently as the positive value.

EXAMPLE 1.—There are two numbers whose sum is 40, and the sum of their squares is 818. What are the numbers?

SOLUTION.—Let $x =$ one number, and $40 - x =$ the other number.

Then, by the conditions, $x^2 + (40 - x)^2 = 818$

whence, $x^2 + 1{,}600 - 80x + x^2 = 818$

Combining, $2x^2 - 80x = -782$

or $x^2 - 40x = -391$

From the formula, $x = \frac{1}{2}\left(40 \pm \sqrt{40^2 - 4 \times 391}\right) = 23 \text{ or } 17$

whence, $x = 23$ or 17

and $40 - x = 17$ or 23 } Ans.

Both answers fulfil the conditions.

EXAMPLE 2.—An iron bar weighs 36 pounds. If it had been 1 foot longer, each foot would have weighed $\frac{1}{2}$ a pound less. Find the length of the bar.

SOLUTION.—Let $x =$ the length of the bar in feet.

Then, $\frac{36}{x} =$ the weight per foot, and

$\frac{36}{x+1} =$ the weight per foot if the bar were 1 foot longer.

By the conditions, $$\frac{36}{x} - \frac{36}{x+1} = \frac{1}{2}$$

Clearing of fractions, $72x + 72 - 72x = x^2 + x$

or $$x^2 + x - 72 = 0$$

By inspection, $x^2 + x - 72 = (x+9)(x-8) = 0$

whence, $x = 8$ ft. or -9 ft. Ans.

PROOF.— $\frac{36}{8} = 4\frac{1}{2}$; $\frac{36}{8+1} = 4$; $4\frac{1}{2} - 4 = \frac{1}{2}$

Or $\frac{36}{-9} = -4$; $\frac{36}{-9+1} = -4\frac{1}{2}$; $-4 - (-4\frac{1}{2}) = \frac{1}{2}$

Only the positive value is required, although both values will satisfy the equation.

EXAMPLE 3.—A number of men ordered a yacht to be built for \$6,300. Each man was to pay the same amount, but two of them withdrew, making it necessary for those remaining to advance \$200 more than they otherwise would have done. How many men were there at first?

SOLUTION.—Let $x =$ the number of men at first.

Then, $\frac{6,300}{x} =$ what each was to have paid, and

$\frac{6,300}{x-2} =$ what each finally paid.

By the conditions, $$\frac{6,300}{x-2} - \frac{6,300}{x} = 200$$

Clearing of fractions and combining,

$$200x^2 - 400x = 12,600$$

or $$x^2 - 2x - 63 = 0$$

By inspection, $x^2 - 2x - 63 = (x-9)(x+7) = 0$

whence, $x = 9$ or -7 Ans.

PROOF.— $\frac{6,300}{9} = 700$; $\frac{6,300}{9-2} = 900$; $900 - 700 = 200$

Or $\frac{6,300}{-7} = -900$; $\frac{6,300}{-7-2} = -700$; $-700 - (-900) = 200$

Only the positive value can be used.

EXAMPLE 4.—A and B start at the same time to travel 150 miles. A travels 3 miles an hour faster than B, and finishes his journey $8\frac{1}{3}$ hours before him. How many miles did each travel per hour?

SOLUTION.—Let x = number of miles A traveled per hour, and
$x-3$ = number of miles B traveled per hour.

Then, $\frac{150}{x}$ = the time in which A performs the journey, and

$\frac{150}{x-3}$ = the time in which B performs the journey.

By the conditions, $$\frac{150}{x-3} - \frac{150}{x} = 8\tfrac{1}{3}$$

Clearing of fractions and combining,

$$25x^2 - 75x = 1{,}350$$

or $$x^2 - 3x - 54 = 0$$

By inspection, $x^2 - 3x - 54 = (x-9)(x+6) = 0$;

whence, $$x = 9 \text{ or } -6$$

and $$x - 3 = 6 \text{ or } -9$$

Using the positive values, A traveled 9 miles per hour and B traveled 6 miles per hour. Ans.

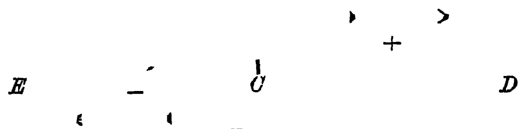


FIG. 1

39. As an illustration of the use of the negative values, consider the following explanation, which refers to the preceding example. In Fig. 1 let C be the starting point. Call any advance in the direction of the upper arrow, or from C toward D, positive, and in the opposite direction, negative. Let E and D be each 150 miles from C. Suppose that a train of cars 150 miles long has one end at C and the other end at D, and that the train is moving in the direction from C to E at the rate of 15 miles per hour. Now, if A and B start toward D, running on the train at the rate of 9 and 6 miles per hour, respectively, while the train moves 15 miles per hour toward E, the rate of travel of A toward D is $9 - 15 = -6$ miles per hour, and of B, $6 - 15 = -9$ miles per hour. It is now evident that A is traveling toward D 3 miles per hour faster than B. When A has

traveled 150 miles, in other words, when he has reached the end of the train, B has reached the point E; he has traveled negatively farther than A, but if he travels to the end of the train, it will take him $8\frac{1}{3}$ hours longer than it did A.

The preceding paragraph is also an illustration of the statement, that of two negative values, the one which has the less value numerically is the greater.

EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES

40. In the third problem in Art. **25**, it was shown how, under certain conditions, more than one unknown quantity *in an example* may be represented *in an equation*, by expressing the value of each quantity in terms of x, thus producing only the *one unknown quantity x in the equation.*

Sometimes, however, each quantity is represented by a different letter, as x, y, or z, in which case, it is necessary to have as many equations as there are unknown quantities, in order to effect a solution. For example, if it were required to find the value of x in the equation $x + y = 10$, x and y being unknown quantities, we should have $x = 10 - y$, x being still undetermined because its value is in terms of the unknown quantity y. There must be another equation, therefore, expressing some other relation between the unknown quantities x and y, in order to fix their values. The equations which fix the values of the unknown quantities must be *independent and simultaneous.*

41. Independent equations are those which express different relations between the unknown quantities. Thus, $x + y = 4$, and $xy = 6$ express different relations between x and y, and are independent. But $x + y = 4$, and $3x + 3y = 12$, are not independent, because, by dividing both members of the second equation by 3, it reduces to the first equation, and thus expresses the *same* relations between the unknown quantities.

42. Simultaneous equations are such as will be satisfied (Art. **18**) by substituting the same values for the same unknown quantities in each equation.

43. Equations containing more than one unknown quantity are solved by so combining them as to obtain a single equation containing but one unknown quantity. This process is called **elimination.** In what follows, equations containing two unknown quantities will be considered.

44. To eliminate by substitution:

Rule.—*From one equation, find the value of one of the unknown quantities in terms of the other. Substitute this value for the same unknown quantity in the other equation.*

EXAMPLE.—Solve the equations

$$2x + 3y = 18 \qquad (1)$$
$$3x - 2y = 1 \qquad (2)$$

SOLUTION.—It will be more convenient to first find the value of x in (2), since, after transposing $-2y$ to the second member, it will become positive.

Transposing $-2y$ in (2), $3x = 1 + 2y$.

Dividing both members by 3,

$$x = \frac{1 + 2y}{3} \qquad (3)$$

This gives the value of x in terms of y.

Substituting this value of x for the x in (1),

$$\frac{2(1 + 2y)}{3} + 3y = 18$$

Removing the parenthesis,

$$\frac{2 + 4y}{3} + 3y = 18$$

Clearing of fractions, $2 + 4y + 9y = 54$

Transposing the 2 and uniting the $4y$ and $9y$,

$$13y = 52$$

whence, $y = 4$ Ans.

Now, having the value of y, we may substitute it for y in any of the above equations containing both x and y, and thus obtain a value for x.

Substituting this value in equation (3),

$$x = \frac{1 + 2 \times 4}{3}$$

whence, $$x = 3 \quad \text{Ans.}$$

45. To eliminate by comparison:

Rule.—*From each equation find the value of one of the unknown quantities in terms of the other. Form a new equation by placing these two values equal to each other and solve.*

Elimination by comparison depends upon the principle that quantities which are equal to the same or two equal quantities are equal to each other. Thus, if $y = 2$ and $x = 2$, y is evidently equal to x.

EXAMPLE.—Solve the same equations as before,

$$2x + 3y = 18 \qquad (1)$$
$$3x - 2y = 1 \qquad (2)$$

SOLUTION.—First obtain the value of x in each equation, it being more convenient to obtain in this case than y.

Transposing $3y$ in (1), $$2x = 18 - 3y$$

or $$x = \frac{18 - 3y}{2} \qquad (3)$$

Transposing $-2y$ in (2), $$3x = 1 + 2y$$

or $$x = \frac{1 + 2y}{3} \qquad (4)$$

Placing the values of x in (3) and (4) equal to each other,

$$\frac{18 - 3y}{2} = \frac{1 + 2y}{3}$$

Clearing of fractions, $$54 - 9y = 2 + 4y$$

Transposing and uniting terms,

$$-13y = -52$$

whence, $$y = 4 \quad \text{Ans.}$$

Substituting this value in (4),

$$x = \frac{1 + 8}{3} = 3 \quad \text{Ans.}$$

46. To eliminate by addition or subtraction:

Rule.—*Select the unknown quantity to be eliminated, and multiply the equations by such numbers as will make the coefficients of this quantity equal in the resulting equations. If the signs of the terms having the same coefficient are alike, subtract one equation from the other; if unlike, add the two equations.*

It is evident that this will not destroy the equality, because adding or subtracting two equations is equivalent to adding the same quantity to, or subtracting it from, both members.

EXAMPLE.—Solve the same equations as before,

$$2x + 3y = 18 \qquad (1)$$
$$3x - 2y = 1 \qquad (2)$$

FIRST SOLUTION.—Since the signs of the terms containing x in each equation are alike, x may be eliminated by subtraction. If the first equation be multiplied by 3 and the second by 2, the coefficients of x in each equation will become equal. Hence,

Multiplying (1) by 3, $6x + 9y = 54$ (3)
Multiplying (2) by 2, $6x - 4y = 2$ (4)
Subtracting (4) from (3), $13y = 52$
whence, $y = 4$ Ans.

Substituting this value of y for the y in (2),

$$3x - 8 = 1$$

Transposing, $3x = 9$
or $x = 3$ Ans.

SECOND SOLUTION.— $2x + 3y = 18$ (1)
$3x - 2y = 1$ (2)

Since the signs of the terms containing y in each equation are unlike, y may be eliminated by addition.

Multiplying (1) by 2, $4x + 6y = 36$ (3)
Multiplying (2) by 3, $9x - 6y = 3$ (4)
Adding (3) and (4), $13x = 39$
whence, $x = 3$ Ans.
Substituting in (1), $6 + 3y = 18$
$3y = 12$
$y = 4$ Ans.

MISCELLANEOUS EXAMPLES

47. From the foregoing it will be seen that any one of the three methods of elimination can be applied to the solution of equations. The student must use his judgment as to which is the best one to apply in any case.

EXAMPLE 1.—Solve the equations

$$\frac{3}{x}+\frac{1}{y}=\frac{5}{4} \qquad (1)$$

$$\frac{2}{x}-\frac{3}{y}=-1 \qquad (2)$$

SOLUTION.—Multiplying (1) by 3,

$$\frac{9}{x}+\frac{3}{y}=\frac{15}{4} \qquad (3)$$

Adding (2) and (3),

$$\frac{11}{x}=\frac{15}{4}-1=\frac{11}{4}$$

Clearing of fractions $\quad 44 = 11x$

or $\quad x = 4$ Ans.

Substituting this value of x in (1),

$$\frac{3}{4}+\frac{1}{y}=\frac{5}{4}$$

Clearing of fractions, $\quad 3y+4 = 5y$

Transposing, $\quad -2y = -4$

or $\quad y = 2$ Ans.

EXAMPLE 2.—Solve the equations

$$x+36y = 900 \qquad (1)$$

$$36x+y = 1{,}320 \qquad (2)$$

SOLUTION.—Adding (1) and (2),

$$37x+37y = 2{,}220 \qquad (3)$$

Dividing by 37, $\quad x+y = 60 \qquad (4)$

Subtracting (4) from (1), $\quad 35y = 840$

$y = 24$ Ans.

Substituting this value in (4),

$$x+24 = 60$$

$$x = 36 \text{ Ans.}$$

EXAMPLE 3.—Solve the equations

$$\frac{m}{x}+\frac{n}{y} = a \qquad (1)$$

$$\frac{n}{x}+\frac{m}{y} = b \qquad (2)$$

SOLUTION.—Multiplying (1) by m,

$$\frac{m^2}{x}+\frac{mn}{y} = am \qquad (3)$$

Multiplying (2) by n, $\quad \frac{n^2}{x}+\frac{mn}{y} = bn \qquad (4)$

Subtracting (4) from (3),

$$\frac{m^2-n^2}{x} = am-bn$$

Clearing of fractions, $m^2-n^2 = (am-bn)x$

whence, $\quad x = \frac{m^2-n^2}{am-bn}$ Ans.

Multiplying (1) by n, $\quad \frac{mn}{x}+\frac{n^2}{y} = an \qquad (5)$

Multiplying (2) by m, $\quad \frac{mn}{x}+\frac{m^2}{y} = bm \qquad (6)$

Subtracting (6) from (5),

$$\frac{n^2-m^2}{y} = an-bm$$

Clearing of fractions, $n^2-m^2 = (an-bm)y$

whence, $\quad y = \frac{n^2-m^2}{an-bm}$, or $\frac{m^2-n^2}{bm-an}$ Ans.

EXAMPLES FOR PRACTICE

Solve the following equations:

1. $\left.\begin{aligned}3x+7y &= 33.\\ 2x+4y &= 20.\end{aligned}\right\}$ Ans. $\begin{cases}x = 4\\ y = 3\end{cases}$

2. $\left.\begin{aligned}8y+12x &= 116.\\ 2x-y &= 3.\end{aligned}\right\}$ Ans. $\begin{cases}x = 5\\ y = 7\end{cases}$

3. $\left.\begin{aligned}ax+by &= m.\\ cx+dy &= n.\end{aligned}\right\}$ Ans. $\begin{cases}x = \dfrac{dm-bn}{ad-bc}\\ y = \dfrac{an-cm}{ad-bc}\end{cases}$

4. $\left.\begin{array}{l}\frac{a}{x}+\frac{b}{y}=m.\\ \frac{c}{x}+\frac{d}{y}=n.\end{array}\right\}$ Ans. $\left\{\begin{array}{l}x=\frac{ad-bc}{dm-bn}\\ y=\frac{bc-ad}{cm-an}\end{array}\right.$

5. $\left.\begin{array}{l}\frac{6}{x}-\frac{3}{y}=4.\\ \frac{8}{x}+\frac{15}{y}=-1.\end{array}\right\}$ Ans. $\left\{\begin{array}{l}x=2\\ y=-3\end{array}\right.$

QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES

48. The methods of solving will be illustrated by the solution of a few examples.

Case I.—*When elimination may be performed by the methods given for simple equations in Arts.* **41 46.**

Example 1.—Solve the equations

$$x^2+y^2=13 \qquad (1)$$
$$x+y=1 \qquad (2)$$

Solution.—Transposing the x in (2),

$$y=1-x \qquad (3)$$

Substituting the value of y in (1),

$$x^2+(1-x)^2=13$$

or

$$x^2+1-2x+x^2=13$$

Transposing and uniting terms,

$$2x^2-2x=12$$

or

$$x^2-x-6=0$$

By inspection, $x^2-x-6=(x-3)(x+2)=0$

whence, $x=3 \text{ or } -2$

Now, two values must be found for y which will satisfy the equations when $x=3$ and $x=-2$.

Substituting these values of x in (3),

$$\left.\begin{array}{l}\text{when } x=3,\ y=-2\\ \text{when } x=-2,\ y=3\end{array}\right\}\ \text{Ans.}$$

This is the form in which answers to simultaneous quadratic equations should always be written.

EXAMPLE 2.—Solve the equations

$$4x^2 - 3y^2 = -11 \qquad (1)$$
$$11x^2 + 5y^2 = 301 \qquad (2)$$

SOLUTION.—Multiplying (1) by 5,

$$20x^2 - 15y^2 = -55 \qquad (3)$$

Multiplying (2) by 3,

$$33x^2 + 15y^2 = 903 \qquad (4)$$

Adding (3) and (4), $53x^2 = 848$

or $x^2 = 16$

Extracting the square root, $x = \pm 4$

Substituting $+4$ for x in (2),

$$11 \times 16 + 5y^2 = 301.$$

or

$$5y^2 = 125$$
$$y^2 = 25$$
$$y = \pm 5$$

Substituting -4 for x in (2) will evidently give the same result, since $(-4)^2 = 16$, the same as 4^2. Hence,

$$\left.\begin{array}{l} \text{when } x = 4, y = \pm 5 \\ \text{when } x = -4, y = \pm 5 \end{array}\right\} \text{ Ans.}$$

49. Case II.—*When the equations may be so combined or reduced as to produce an equation having for the first member an expression of the form $x^2 + 2xy + y^2$ or $x^2 - 2xy + y^2$.*

No rule can be given for solving examples under this case. The student must depend upon his own ingenuity.

EXAMPLE 1.—Solve the equations

$$x^2 + y^2 = 25 \qquad (1)$$
$$xy = 12 \qquad (2)$$

SOLUTION.—Multiplying (2) by 2,

$$2xy = 24 \qquad (3)$$

Adding (1) and (3), $x^2 + 2xy + y^2 = 49 \qquad (4)$

Subtracting (3) from (1),

$$x^2 - 2xy + y^2 = 1 \qquad (5)$$

Extracting the square root of both terms of (4), see Art. **20**, Part 2,

$$x + y = \pm 7 \qquad (6)$$

Extracting the square root of both terms of (5),

$$x - y = \pm 1 \qquad (7)$$

This gives two simple equations, from which either x or y may be eliminated by addition or subtraction. Adding (6) and (7), the first member of the new equation will be $2x$, and the second member may have four values as follows:

$$7+1,\ 7-1,\ -7+1 \text{ or } -7-1$$

or $$2x = 8,\ 6,\ -6 \text{ or } -8$$

whence, $$x = 4,\ 3,\ -3 \text{ or } -4$$

By substituting these values in (2) we have for the corresponding values of y, $y = 3, 4, -4$, or -3.

These values may also be obtained by subtracting (7) from (6). The answers would be written,

$$\left.\begin{array}{r} \text{when} \quad x = 4, y = 3;\ x = 3, y = 4 \\ x = -3, y = -4;\ x = -4, y = -3 \end{array}\right\} \text{Ans.}$$

NOTE.—In solving examples under this case, the object is always to produce two equations, one with $x+y$ and one with $x-y$ for the first member, from which the value of x or y can easily be found.

EXAMPLE 2.—Solve the equations

$$x^3+y^3 = 133 \qquad (1)$$
$$x^2-xy+y^2 = 19 \qquad (2)$$

SOLUTION.— x^3+y^3 is divisible by $x+y$ (see Art. **32**, Part 2); hence, $x^3+y^3 = (x+y)(x^2-xy+y^2) = 133$. Divide both members of the equation by $x+y$.

$$x^2-xy+y^2 = \frac{133}{x+y}$$

Therefore, $$\frac{133}{x+y} = 19$$

and $$x+y = \tfrac{133}{19} = 7 \qquad (3)$$

This gives at once an equation with $x+y$ for the first member. To obtain a value for $x-y$, it will be noticed that the first member of (2) lacks only one $-xy$ of being $x^2-2xy+y^2$, from which $x-y$ may be obtained; hence, proceed to obtain a value for $-xy$, to add to (2).

Squaring (3), $$x^2+2xy+y^2 = 49 \qquad (4)$$
Writing (2) under (4), $$x^2 \quad -xy+y^2 = 19$$

and subtracting, $$3xy = 30$$
or $$xy = 10 \qquad (5)$$

Subtracting (5) from (2),

$$x^2-2xy+y^2 = 9$$

Extracting the square root (see Art. **20**, Part 2),

$$x-y = \pm 3 \qquad (6)$$

Adding (6) and (3), $$2x = 10 \text{ or } 4$$
$$x = 5 \text{ or } 2$$

Subtracting (6) from (3), $$2y = 4 \text{ or } 10$$
$$y = 2 \text{ or } 5$$

Or, solving (5) for x, $$x = \frac{10}{y}$$

Substituting the value of x in (3),

$$\frac{10}{y} + y = 7$$

Clearing of fractions and changing signs,

$$y^2 - 7y = -10$$

Solving for y, $$y = 5 \text{ or } 2$$

Substituting their values in (3),

$$x = 2 \text{ or } 5$$

Hence, when $$\left.\begin{array}{l} x = 5, y = 2 \\ x = 2, y = 5 \end{array}\right\} \text{Ans.}$$

EXAMPLES FOR PRACTICE

Solve the following equations:

1. $\left.\begin{array}{l} x^2 + y^2 = 29. \\ x + y = 3. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x = 5,\ y = -2 \\ x = -2,\ y = 5 \end{array}\right.$

2. $\left.\begin{array}{l} 2x^2 + y^2 = 9. \\ 5x^2 + 6y^2 = 26. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x = 2, y = \pm 1, \\ x = -2, y = \pm 1 \end{array}\right.$

3. $\left.\begin{array}{l} x + y = -1. \\ xy = -56. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x = 7,\ y = -8 \\ x = -8,\ y = 7 \end{array}\right.$

PROBLEMS LEADING TO EQUATIONS WITH TWO UNKNOWN QUANTITIES

50. A few examples involving equations with two unknown quantities will now be given. The student should pay particular attention to the manner in which the equations are formed from the conditions given.

EXAMPLE 1.—A certain fraction becomes equal to $\frac{1}{2}$ if 3 is added to its numerator, and equal to $\frac{2}{7}$ if 3 is added to its denominator. What is the fraction?

SOLUTION.—Let $\frac{x}{y}$ = the required fraction.

By the conditions, $$\frac{x+3}{y} = \frac{1}{2}$$

and $$\frac{x}{y+3} = \frac{2}{7}$$

Solving these equations, $x = 6$ and $y = 18$
That is, the fraction is $\frac{6}{18}$. Ans.

EXAMPLE 2.—A crew can row 20 miles in 2 hours down stream, and 12 miles in 3 hours up stream. Required, the rate per hour of the current, and the rate per hour at which the crew would row in still water.

Let x = rate per hour of crew in still water
and y = rate per hour of current.

Then, $x+y$ = rate per hour rowing down stream
and $x-y$ = rate per hour rowing up stream.

Since they row 20 miles in two hours down stream, in one hour, they would row $\frac{20}{2} = 10$ miles, or at the rate of 10 miles per hour. Also, in rowing up stream, they would row at the rate of $\frac{12}{3} = 4$ miles per hour.

Consequently, $x+y = 10$ (1)
$x-y = 4$ (2)
Adding, $2x = 14$
or $x = 7$
Subtracting, $2y = 6$
or $y = 3$

Hence, the rate of the crew is 7 miles per hour, and of the current, 3 miles per hour. Ans.

EXAMPLE 3.—A wine merchant has two kinds of wine, which cost 72 cents and 40 cents a quart, respectively. How much of each must he take to make a mixture of 50 quarts worth 60 cents a quart?

SOLUTION.—Let x = required number of quarts at 72 cents
and y = required number of quarts at 40 cents.

Then, $72x$ = cost in cents of the first kind,
$40y$ = cost in cents of the second kind
and $60 \times 50 = 3{,}000$ = cost in cents of the mixture.

By the conditions, $x+y = 50$
and $72x + 40y = 3{,}000$

Solving, $x = 31\frac{1}{4}$ qt. and $y = 18\frac{3}{4}$ qt. Ans.

NOTE.—The equation, $x^2 \pm px = \pm q$ is equivalent to the four equations

$$\begin{array}{llr}
x^2 + px = \quad q, \text{ or } x^2 + px - q = 0 & & (1) \\
x^2 - px = \quad q, \text{ or } x^2 - px - q = 0 & & (2) \\
x^2 + px = -q, \text{ or } x^2 + px + q = 0 & & (3) \\
x^2 - px = -q, \text{ or } x^2 - px + q = 0 & & (4)
\end{array}$$

EQUATION (1).—By Art. **32**, the roots of equation (1) are $-\frac{1}{2}(p + \sqrt{p^2 + 4q})$ and $-\frac{1}{2}(p - \sqrt{p^2 + 4q})$. Applying the principle of Art. **33**, $[x + \frac{1}{2}(p + \sqrt{p^2 + 4q})] \times [x + \frac{1}{2}(p - \sqrt{p^2 + 4q})] = x^2 + px - q$, as here shown by actual multiplication.

$$\begin{array}{llllll}
x + \frac{1}{2}p + \frac{1}{2}\sqrt{p^2 + 4q} & & & & & \\
x + \frac{1}{2}p - \frac{1}{2}\sqrt{p^2 + 4q} & & & & & \\
x^2 + \frac{1}{2}px & + \frac{1}{2}x\sqrt{p^2 + 4q} & & & & \\
\quad + \frac{1}{2}px & & + \frac{1}{4}p^2 & + \frac{1}{4}p\sqrt{p^2 + 4q} & & \\
 & - \frac{1}{2}x\sqrt{p^2 + 4q} & & - \frac{1}{4}p\sqrt{p^2 + 4q} & - \frac{1}{4}p^2 - q & \\
x^2 + px & \quad 0 & + \frac{1}{4}p^2 & \quad 0 & - \frac{1}{4}p^2 - q & \\
\text{or } x^2 + px - q & & & & &
\end{array}$$

REMARK.— $\frac{1}{2}\sqrt{p^2 + 4q} \times -\frac{1}{2}\sqrt{p^2 + 4q} = \frac{1}{2}(p^2 + 4q)^{\frac{1}{2}} \times -\frac{1}{2}(p^2 + 4q)^{\frac{1}{2}} = -\frac{1}{4}(p^2 + 4q) = -\frac{1}{4}p^2 - q$.

EQUATION (2).—The roots of equation (2) are $\frac{1}{2}(p + \sqrt{p^2 + 4q})$ and $\frac{1}{2}(p^2 - \sqrt{p^2 + 4q})$, and $[x - \frac{1}{2}(p + \sqrt{p^2 + 4q})] \times [x - \frac{1}{2}(p - \sqrt{p^2 + 4q})] = x^2 - px - q$.

$$\begin{array}{llllll}
x - \frac{1}{2}p - \frac{1}{2}\sqrt{p^2 + 4q} & & & & & \\
x - \frac{1}{2}p + \frac{1}{2}\sqrt{p^2 + 4q} & & & & & \\
x^2 - \frac{1}{2}px & - \frac{1}{2}x\sqrt{p^2 + 4q} & & & & \\
\quad - \frac{1}{2}px & & + \frac{1}{4}p^2 & + \frac{1}{4}p\sqrt{p^2 + 4q} & & \\
 & + \frac{1}{2}x\sqrt{p^2 + 4q} & & - \frac{1}{4}p\sqrt{p^2 + 4q} & - \frac{1}{4}p^2 - q & \\
x^2 - px & \quad 0 & + \frac{1}{4}p^2 & \quad 0 & - \frac{1}{4}p^2 - q & \\
\text{or } x^2 - px - q & & & & &
\end{array}$$

EQUATIONS (3) AND (4).—Equations (3) and (4) can be produced in the same manner, by multiplying its roots, which are: for (3), $-\frac{1}{2}(p + \sqrt{p^2 - 4q})$ and $-\frac{1}{2}(p - \sqrt{p^2 - 4q})$; for (4), $\frac{1}{2}(p + \sqrt{p^2 - 4q})$ and $\frac{1}{2}(p - \sqrt{p^2 - 4q})$. Since these are all the cases that can arise, the formula of Art. **32** is correct.

GEOMETRICAL DRAWING

(PART 1)

PRINCIPLES

INTRODUCTION

1. Drawing is the art of representing objects or ideas on a plane surface, as on a sheet of paper. Drawings are of two classes, according to the plan followed in making them. When drawings are made free hand, that is, without the use of instruments, they are **freehand drawings**; when they are made with the aid of instruments they are **mechanical drawings**, or, as they are sometimes called, *instrumental drawings*. The use of instruments results in greater accuracy than can be secured by the hand alone.

Geometrical drawings are mechanical drawings for which the *positions* of lines must be determined from the principles of geometry. Geometry is the branch of mathematics that treats of space and its relations; it is the science of the relative positions of points, lines, angles, surfaces, and solids. Geometrical processes are based on *reason* independent of measurement, construction, etc. Geometrical principles are *absolutely true*, and if one understands these principles, the correctness of a drawing will depend entirely on accuracy in the use of drawing instruments. **Mensuration** is the branch of mathematics that has to do with finding lengths of lines, areas of surfaces, and volumes of solids.

2. In order that those who have no knowledge of geometry and mensuration can proceed intelligently with the drawing

work, definitions and explanations of such terms and propositions as will be encountered are given in the following pages. These definitions are important and must be thoroughly understood.

DEFINITIONS

LINES

3. A **point** indicates position only; it has no length, breadth, or thickness. It may be represented on paper by a dot.

4. A **line** has one dimension only, namely, *length;* it may be straight, curved, or irregular.

5. If the direction of a line does not change, it is a **straight line,** as in Fig. 1 (*a*); if the direction changes continually about

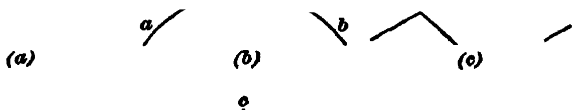


FIG. 1

a point, or center, it is a **curved line,** as in Fig. 1 (*b*), where *a b* is the line and *c* the center from which it was drawn. A line made up of straight lines having different directions, as in Fig. 1 (*c*), is called an **irregular line.**

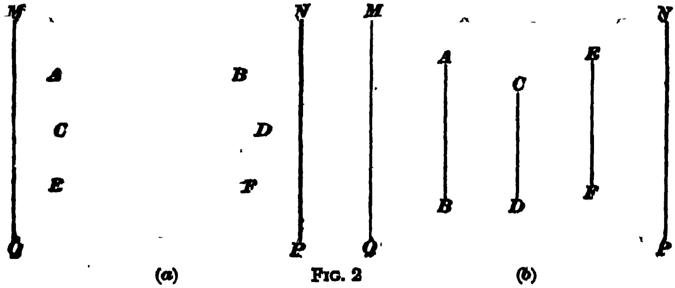


FIG. 2

6. A **horizontal line,** as the term is used in drawing, is a straight line running from left to right, or from right to left, without slanting either upwards or downwards. In Fig. 2 (*a*),

A B, *C D*, *E F* represent horizontal lines on a sheet of drawing paper *M N P Q*. These lines are parallel to the top and bottom edges of the sheet.

7. Vertical lines with reference to the drawing paper *M N P Q*, Fig. 2 (*b*), are the straight lines *A B*, *C D*, *E F* running exactly up and down, without slanting to either the right or the left.

Parallel lines are those that are at an equal distance apart throughout their entire length, as the horizontal lines in Fig. 2 (*a*) and the vertical lines in Fig. 2 (*b*).

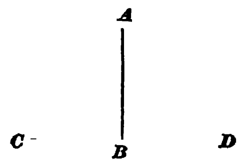


FIG. 3

8. A line is **perpendicular** to another line when it meets it so as not to incline toward it on either side; for example, in Fig. 3, line *A B* is perpendicular to the horizontal line *C D*; however, for a line to be perpendicular to another, it is not necessary that either line should be horizontal or vertical.

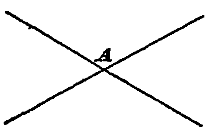


FIG. 4

9. Oblique lines are lines that are neither vertical nor horizontal, as in Fig. 4. Such lines are frequently referred to as **inclined lines.**

10. When two lines cross, or cut, each other as in Fig. 4, they are said to **intersect.** The point *A* at which they meet is the **point of intersection.**

ANGLES

11. An **angle,** Fig. 5 (*a*), is the opening between two straight lines that meet in a point. The two straight lines are the **sides,** and the point where the lines meet is the **vertex** of the angle. Thus, the straight lines *O A* and *O B* form an angle at the point *O*; the lines *O A* and *O B* are the sides of this angle, and the point *O* is its vertex.

An angle is usually referred to by naming a letter on each of its sides and a third letter at the vertex, the letter at the

Polygons are named according to the number of their sides. A **triangle** has three sides; a **quadrilateral,** four; a **pentagon,** five; a **hexagon,** six; a **heptagon,** seven; an **octagon,** eight; and a **decagon,** ten. These polygons are illustrated in Fig. 8.

A **regular polygon** is one in which all the sides and all the angles are equal. The triangle shown in Fig. 8 is a regular polygon because all its angles and all its sides are equal.

The **perimeter** of a polygon is the distance around its sides.

TRIANGLES

18. Triangles have three sides and three angles; they are named according to their sides as *isosceles, equilateral,* and

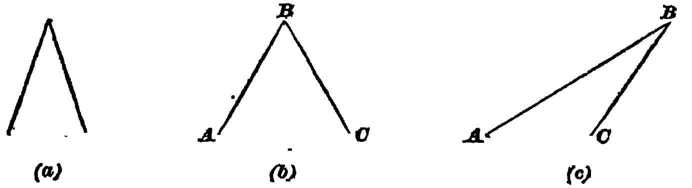


FIG. 9

scalene triangles; and according to their angles, as *right-angled* and *oblique-angled* triangles.

19. An **isosceles triangle,** Fig. 9 (*a*), is one having two of its sides equal. An **equilateral triangle,** Fig. 9 (*b*), is one having all three of its sides equal. A **scalene triangle,**

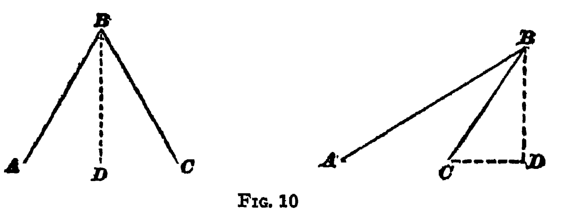


FIG. 10

Fig. 9 (*c*) is one having no two sides equal. An **oblique triangle** is one having no right angle.

20. The base of any triangle is the side on which the triangle is considered to stand; any side may be considered as the base. In Fig. 9 (*b*) and (*c*), the line *A C* is the base.

21. The **altitude**, or *height*, of a triangle is the vertical distance between the vertex of the angle opposite the base and the base, or the base produced. In Fig. 10 (*a*) and (*b*), *B* is the vertex of the angle formed by intersection of the sides *A B* and *B C*; the side *A C* is the base, and *B D* is the altitude. In (*b*), *A D* is the base produced, or extended.

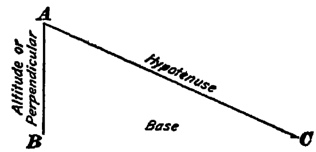


Fig. 11

22. A **right-angled triangle**, or *right triangle*, is a triangle having one right angle. In Fig. 11, the sides *A B* and *B C* form a right angle at *B*. The side opposite the right angle, as *A C*, is the **hypotenuse**.

QUADRILATERALS

23. **Parallelograms** are quadrilaterals whose opposite sides are parallel and opposite angles are equal. There are

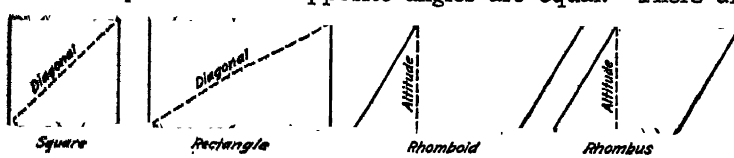


Fig. 12

four kinds of parallelograms: the *square*, the *rectangle*, the *rhomboid*, and the *rhombus*, shown, respectively, in Fig. 12.

24. A **diagonal** of any parallelogram is a straight line joining opposite corners, as shown at the left in Figs. 12 and 13.

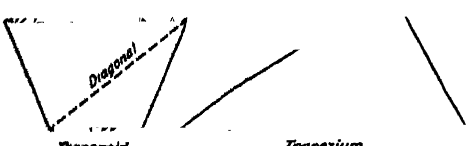


Fig. 13

25. A **quadrilateral** with only two sides parallel is a **trapezoid**, and one with no two sides parallel is called a **trapezium**. These are shown in Fig. 13.

If, as in Fig. 17, two lines *o c* and *o d* are drawn from the center *o* of a circle and cutting its circumference, the number of degrees on that part of the circumference lying between the two lines is the measure of the angle between the lines.

In Fig. 17, one-fourth of the circle is divided into spaces of 5 degrees each, and it is seen that the arc *a b* included between the lines *o c* and *o d* contains seven of these spaces, or 35 degrees.

As the lines *o c* and *o d* are extended past the points *a* and *b*, the distance between them constantly increases, and when they cut the arc *c d*, which is part of the circumference of a larger circle having the same center *o*, the distance between them is much greater than where they cut the arc *a b*, but it is the same proportionate part of the circumference of the greater circle; that is, $\frac{35}{360}$, or 35 degrees. Therefore, the measure of the angle *c o d* is the same whether measured on the arc of the large or the small circle.

The **measure of any angle** is the number of degrees on the arc included between its two sides, the arc having the vertex of the angle as its center.

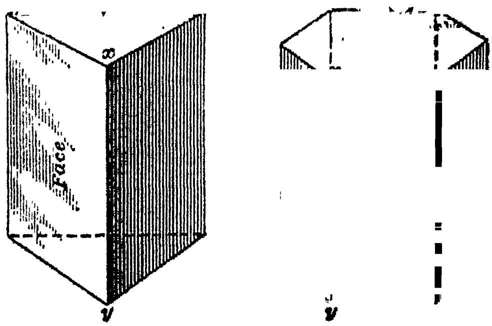
For ordinary drawing work, degree and half-degree divisions give sufficient accuracy of measurement; but for extremely accurate mathematical work degrees are divided into 60ths, which are called minutes, and minutes are divided into 60ths, which are called seconds.

In drawing practice, angles are measured or conveniently laid off by the use of a **protractor**, which is a graduated instrument showing degrees and one-half degrees. This drafting tool will be explained fully further on. The symbol for degrees is $^\circ$, for minutes is $'$, and for seconds is $''$; fifteen degrees twenty minutes and thirty seconds may be written $15^\circ\ 20'\ 30''$.

SOLIDS

31. A **solid** is a body having three dimensions: length, breadth, and thickness. The more common forms of solids met with in engineering and drawing work are **prisms, cylinders, cones, pyramids,** and **spheres.**

32. Prisms and Cylinders.—A **prism** is a body whose ends are equal parallel polygons and whose sides are parallelograms, as shown in Fig. 18 (*a*) and (*b*). Prisms take their names from the shapes of their bases, or ends. Thus, in Fig. 18 (*a*) the base of the prism is a triangle, and in (*b*) a hexagon, and the prisms are called, respectively, **triangular** and **hexagonal** prisms.

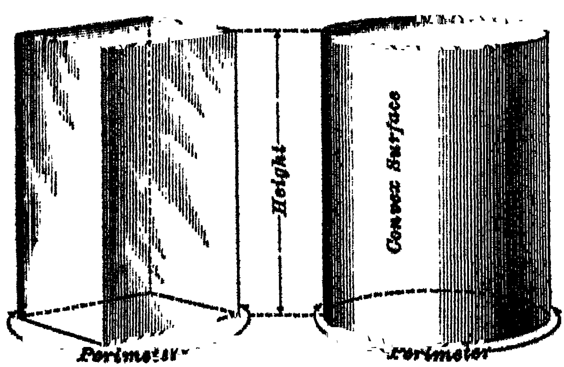


(*a*) FIG. 18 (*b*)

The intersections of the sides, or faces, of prisms, as $x\,y$, are called the edges. A **rectangular** prism is one whose ends are rectangles, as Fig. 19 (*a*). A **cube** is a prism whose ends and faces are square.

A **right prism** is one whose sides are perpendicular to its base, as shown in Fig. 18 and Fig. 19 (*a*). The **entire surface** of a solid is the area of all its sides and ends.

33. A **cylinder** is a round body of uniform diameter with



(*a*) FIG. 19 (*b*)

circles for its ends, as shown in Fig. 19 (*b*). The curved surface of a cylinder is called its **convex surface**.

The **altitude, or vertical height,** of a prism or cylinder is the perpendicular distance between its two ends, as indicated in Fig. 19.

The **perimeter** of a prism or a cylinder is the distance around its base.

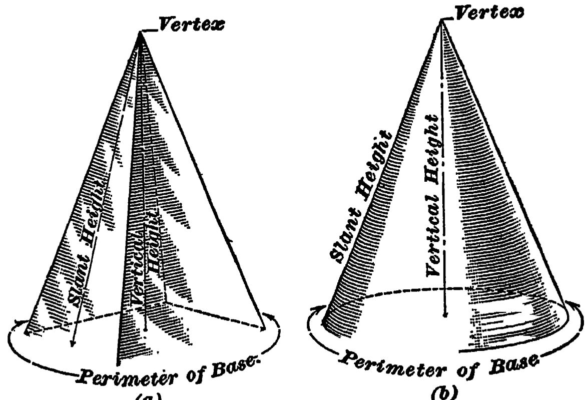


FIG. 20

34. Pyramids and Cones.—A **pyramid** is a solid whose base is a polygon, and whose sides are triangles meeting in a common point, or **vertex,** as shown in Fig. 20 (*a*). A **cone,**

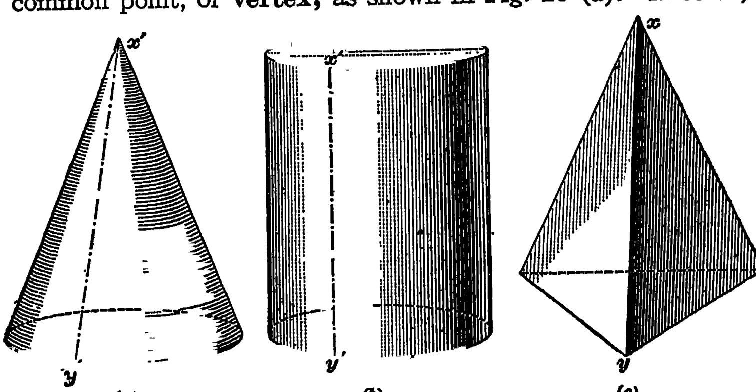


FIG. 21

Fig. 20 (*b*), is a solid with a circular base, and whose convex surface tapers uniformly to a point called the vertex. The

altitude, or vertical height, of a pyramid or a cone is the perpendicular distance from the vertex to the base. The **slant**

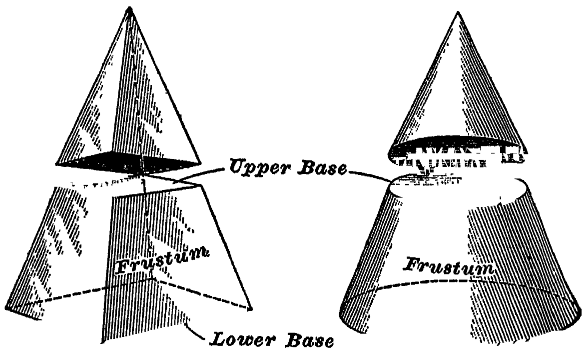


FIG. 22

height of a pyramid is a line drawn from the vertex perpendicular to one of the sides of the base.

The slant height of a cone is measured on a straight line drawn from the vertex to the circumference of the base.

35. In the convex surfaces of cones and cylinders imaginary lines called **elements** are assumed to be drawn. These lines are often referred to in the problems relating to the cone and cylinder. The lines x' y' in Fig. 21 (a) and (b) are elements. The line x y in (c) is an intersection line, or edge line, and is not an element.

36. Frustums of Pyramids or Cones.—When a pyramid or a cone is cut by a plane parallel to its base, as in Fig. 22, the lower part is called the **frustum** of the pyramid or the frustum of the cone, as the case may be.

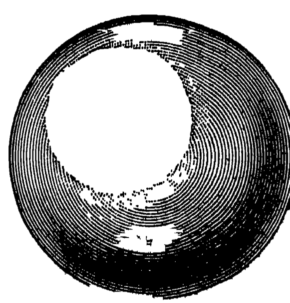


FIG. 23

37. The Sphere.—A **sphere**, Fig. 23, is a solid bounded by a uniformly curved surface, every point of which is equally distant from a point within, called its center. The term **ball** is often used instead of sphere.

DRAWING INSTRUMENTS AND THEIR USE

DRAWING BOARD

38. The drawing board should be made of well-seasoned, straight-grained pine, the grain running lengthwise. To avoid the tendency of the board to warp, the wood of which it is made is thoroughly dried before it is put together. But as wood absorbs moisture from the air, warping will not be avoided if the board is kept near a stove, radiator, or other source of heat. The tendency to warp can be counteracted to a considerable extent by the manner of constructing the board.

A board similar to the one shown in Fig. 24, which is about 21 inches long and 16 inches wide, can be recommended as meeting the requirements of all work in this Section. In the

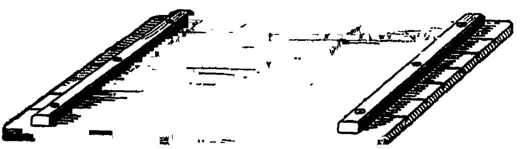


FIG. 24

making of such a board seasoned pine strips of the number necessary to make the required width of board are glued together. Two hardwood cleats are screwed to the back, to prevent the board from bending or warping, and as a further precaution grooves are cut on the back between the wooden strips in the direction of their length. These grooves reduce the tendency of the board to warp and do not materially affect its longitudinal strength. The cleats also raise the board from the table, thus making it easier to change the board to any required position.

The drawing board should be so arranged that the draftsman can do his work conveniently and to the best advantage.

If no drawing table is used, the best support for the board is a solid table of the form shown in Fig. 25. Usually, when in use, the board is placed so that one of its short edges is at the left of the draftsman and the board is inclined as shown. A block of wood or anything else may be used to tilt the board to the desired angle.

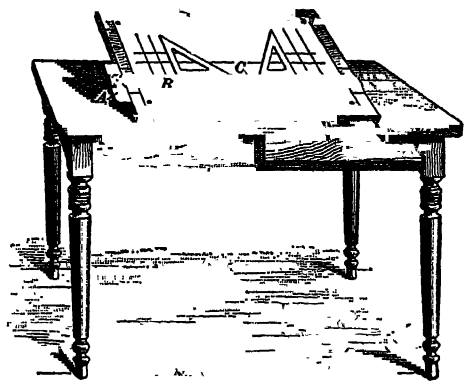


FIG. 25

T SQUARE

39. For drawing horizontal straight lines, the **T square** shown in Fig. 25 is used in the position shown. The head *A* of the T square should be firmly held against the left edge of the drawing board, whether the short or the long side of the board is at the left. The T square must never be used against more than one edge of the board for the same drawing. In some of the geometrical drawing work required, it will be found that the board must be turned with the long side to the left. To draw a horizontal straight line, the blade *B* must be moved so that its edge *C* will be at the place where the line is to be drawn. Any number of parallel straight lines, as shown in Fig. 25, can be drawn by moving the blade the required distance.

DRAWING PAPER

40. The quality of drawing paper recommended for this series of drawing lessons is **cold-pressed demy,** a grade of paper of even grain and rather rough surface. It takes ink well and withstands considerable erasing.

41. Fastening Drawing Paper.—The paper is fastened on the drawing board by means of **thumbtacks,** Fig. 26

which are small tacks having a sharp point and a large flat head. When fastening a sheet of drawing paper on the drawing board, care must be taken to stretch it evenly, so that it will have no wrinkles. To do this, proceed as follows: Lay the paper on the drawing board with the edges parallel to, and equally distant from, the sides. Insert a thumbtack in the upper right-hand corner, about $\frac{1}{4}$ inch from the edge of the paper, and press it in until the head bears evenly on the paper all around. Line the upper edge of the paper so that it is parallel with the ruling edge of the **T**-square blade. Then pull the paper by sliding the hand lightly and diagonally toward the lower left-hand corner, and, holding the paper there, press in another thumbtack, as before. Lay the left hand on the middle of the sheet, slide it very lightly toward the upper left-hand corner, and insert another tack. The fourth tack is inserted in the same way as the third, except that the left hand is slid from the center to the lower right-hand corner. If the paper is wrinkled or loose, it shows that it has been unevenly stretched, and the preceding operation must be repeated until the sheet lies flat and smooth on the board.

FIG. 26

42. In damp weather the paper usually swells and becomes loose and wavy. In such cases a tack may be put in the middle of each edge of the sheet, after the paper has been gently and evenly smoothed from the center to the middle of each edge. The tacks in the corners are then taken out and reinserted a little to one side of their former positions, after the sheet is evenly stretched toward each corner. By putting the four tacks in the middle of the sides first, the drawing will be kept in the same position on the board. This precaution is very important when a **T** square is used.

DRAWING PENCILS

43. For drawing instrumental problems in pencil, a 4H pencil of any good make should be employed. The use of a pencil softer than this is not recommended, as the point of a soft pencil wears away so fast that accurate work cannot be done, and soft black marks rub off and soil the drawing.

The pencil should be sharpened as shown in Fig. 27. Cut the wood away so as to leave about $\frac{1}{4}$ or $\frac{3}{8}$ of an inch of the lead projecting; then sharpen it flat by rubbing it against a fine file or a piece of fine emery cloth or sandpaper that has been fastened to a flat stick. Grind it to a sharp edge like a knife blade, and round the corners very slightly, as shown in the figure. If sharpened to a round point, Fig. 28, the lead will wear away very quickly and make broad lines, and unless the pencil is frequently sharpened it is difficult to draw a line exactly through a point.

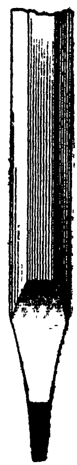


FIG. 27

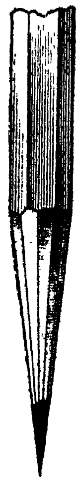


FIG. 28

TRIANGLES

44. Triangles, or set squares, are used for drawing perpendiculars, angles, and parallel lines. The triangles most generally used are shown in Figs. 29 and 30. Each has one right angle, or 90° angle. The triangle shown in Fig. 29 has two angles of 45° each, and that in Fig. 30 one of 60° and one of 30°. They are called *45° and 60° triangles*, respectively.

To draw a vertical line, place the T square in position to draw a horizontal line, and lay the triangle against it, so as to form a right angle. Hold both T square and triangle lightly with the left hand, so as to keep them from slipping, and draw the line with the pen or pencil held in the right hand, and against the

edge of the triangle. Fig. 31 shows the triangles and **T** square in position.

45. To draw parallel lines that are neither vertical nor horizontal.

When the lines are near together, the best way is to place one edge of a triangle, as *a b*, Fig. 32, on the given line *c d*,

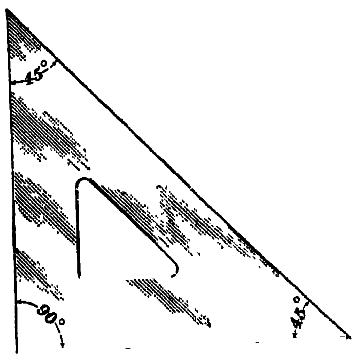


FIG. 29

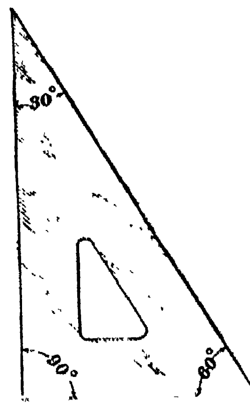


FIG. 30

and lay the other triangle, as *B*, against one of the two edges, holding it fast by letting the left hand rest on it; then move the triangle *A* along the edge of *B*. The edge *a b* will remain parallel to the line *c d*; and when the edge *a b* reaches a point, as *g*, through which it is desired to draw the parallel line, hold both triangles stationary with the left hand and draw the line *e f* by passing the pencil along the edge *a b*. Should the triangle *A* extend so far beyond the edge of the triangle *B* that the desired position cannot be reached, hold *A* stationary with the left hand and shift *B* along the edge of *A* with the right hand and then proceed as before.

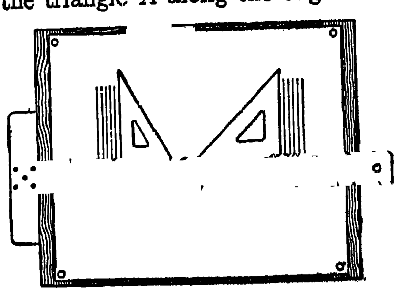


FIG. 31

46. To draw a line at right angles to another line which is neither vertical nor horizontal.

Let $c\,d$, Fig. 33, be the given line (shown at the left-hand

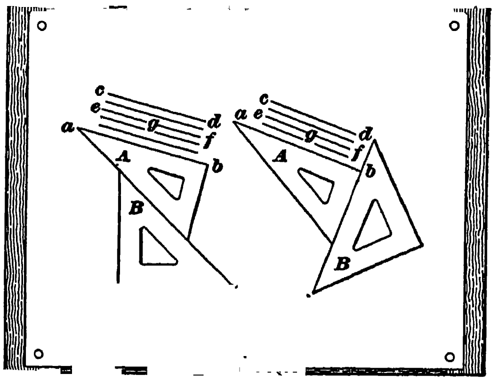


FIG. 32

side). Place one of the shorter edges, as $a\,b$, of the triangle B so that it will coincide with the line $c\,d$; then, keeping the tri-

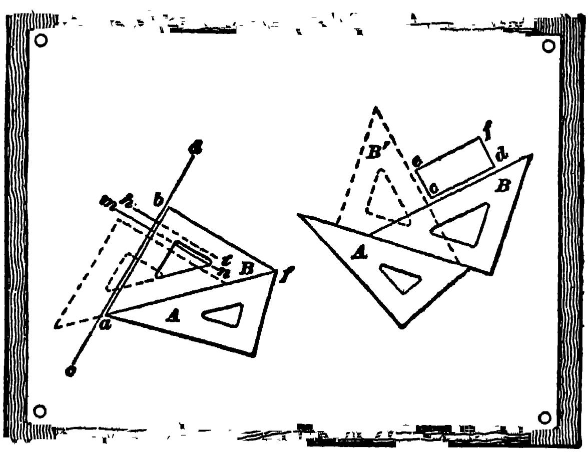


FIG. 33

angle in this position, place the triangle A so that its long edge will come against the long edge of B. Now, holding A securely in place with the left hand, slide B along the edge of A with

the right hand, and draw the lines *h i*, *m n*, etc. perpendicular to *c d* along the edge *b f* of the triangle *B*. The dotted lines show the position of the triangle *B* when moved along the edge of *A*.

47. The right-hand portion of Fig. 33 shows another method of accomplishing the same result, and illustrates how the triangles may be used for drawing a rectangular figure when the sides of the figure make an angle with the **T** square such that the latter cannot be used.

Let the side *c d* of the figure be given. Place the *long* side of the triangle *B* so as to coincide with the line *c d*, and bring the triangle *A* into position against the lower side of *B*, as shown. Now, holding the triangle *A* in place with the left hand, revolve *B* so that its other short edge will rest against the long edge of *A*, as shown in the dotted position at *B'*. The parallel lines *c e* and *d f* may now be drawn through the points *c* and *d* by sliding the triangle *B* on the triangle *A*, as described in connection with Fig. 32. Measure off the required width of the figure on the line *c e*, reverse the triangle *B* again to its original position, still holding the triangle *A* in a fixed position with the left hand, and slide *B* on *A* until the long edge of *B* passes through *e*. Draw the line *e f* through the point *e*, and *e f* will be parallel to *c d*. It is advisable to practice drawing lines that are parallel and perpendicular to each other with the triangles before beginning the drawing lessons that follow.

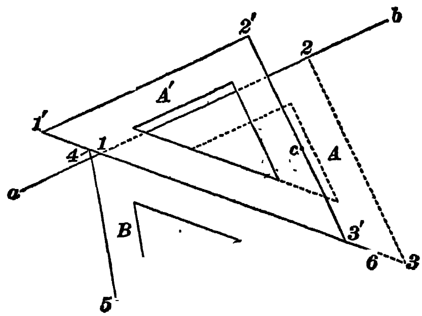


FIG. 34

48. To draw, by means of the triangles, a perpendicular to a line through a given point.

In Fig. 34, *a b* is the given line to which it is required to draw a perpendicular through the point *c*. Place one triangle, as *A*, so that one leg, as *1–2*, lies on the given line, while

the other leg, as *2–3*, is a little to one side of the given point *c*, as shown by the dotted triangle. Place the other triangle *B* along the hypotenuse *1–3* of triangle *A*, as shown by the triangle *4–5–6*. Now slide the triangle *A* along the side of *B* until the leg *2–3* lies on the point through which the perpendicular is to be drawn, as shown by the triangle *1′–2′–3′*. A line drawn with the leg *2′–3′* as a ruling edge will be the required perpendicular.

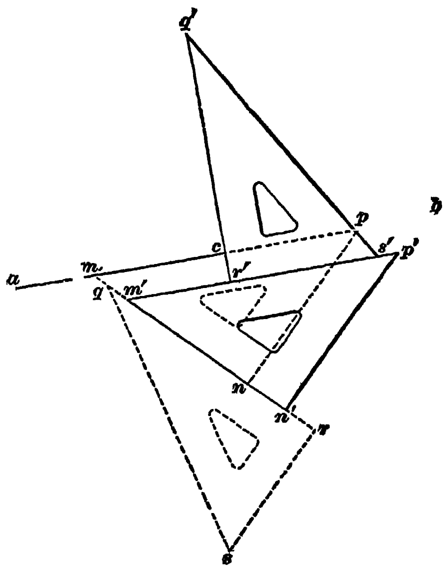


FIG. 35

49. A method that can be used when it is desired to draw a longer line than can be made by the preceding method is as follows:

Let *a b*, Fig. 35, be the given line, and *c* the given point, which, in this problem, lies on the line. First, place one triangle *m p n* so that the hypotenuse *m p* will coincide with the line. Place one edge of the other triangle *q r s* along *m n*, as shown. Hold triangle *q r s* firmly in place and slide triangle *m n p* along *q r* a short distance, until it takes the position *m′ n′ p′*. Hold *m′ n′ p′* firmly, and take the other triangle *q r s* and put it in the position *q′ r′ s′*, with the short side in contact with *m′ p′*. A line drawn along *q′ r′* will be perpendicular to *a b*, and, by sliding triangle *q′ r′ s′* along *m′ p′*, the edge *q′ r′* may be made to pass through any required point.

50. To draw, by means of the triangles, lines making angles of 30°, 45°, or 60° with a given line.

There are several ways of doing this, all similar to those used for the drawing of perpendiculars. The following is one of the convenient methods:

Let $a\,b$, Fig. 36, be the line with which the required line is to make an angle of 30°. Place the 45° triangle so that one of its edges (preferably the hypotenuse) will coincide with $a\,b$, and then slide it down along the edge of the other triangle until it takes the position $m\,n\,p$. Holding it firmly in this position, place the other triangle as shown by $q'\,r'\,s'$, that is, with the vertex of the right angle and that of the 30° angle against $m\,n$. By sliding $q'\,r'\,s'$ along $m\,n$, it is possible to make $q'\,s'$ pass through any required point. Any line drawn along $q'\,s'$, in any of the positions of triangle $q'\,r'\,s'$, will make an angle of 30° with $a\,b$. If it is desired to have the angle opening toward

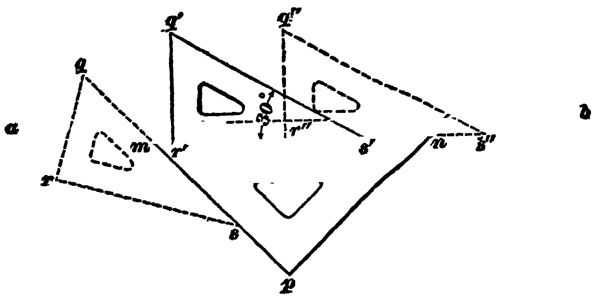


FIG. 36

the right, triangle $q'\,r'\,s'$ must be turned over about $q'\,r'$; in other words, it should be so placed on $m\,n$ that r' will be at the right of s'.

An angle of 60° may be drawn in a similar manner by placing the triangle $q'\,r'\,s'$ so that r' and q' will be on $m\,n$. The method of drawing a 45° angle is the same, but in this case the 60° triangle will have to be used in the position $m\,n\,p$, and the 45° triangle in the position $q'\,r'\,s'$.

In using triangles, it is advisable not to allow one to extend more than half its length beyond the other, otherwise there will not be sufficient bearing surface and there will be a tendency for the triangles to rock, thereby making the work inaccurate.

51. To divide the circumference of a circle into two, four, or eight equal parts.

Any straight line, as $a\,b$, Fig. 37, drawn through the center o of a circle, will divide the circumference into halves. Another

line, as $c\,d$, drawn perpendicular to the first, will divide the circumference into fourths. Two other lines, as $g\,h$ and $e\,f$, drawn through the center at an angle of 45° with $a\,b$, will divide the circumference into eighths. Each of the arcs $a\,g$, $g\,c$, etc. will be one-eighth of the circumference.

52. To divide the circumference of a circle into six equal parts.

There are two methods:

The first method is to open the dividers equal to the radius of the circle and, beginning at any point on the circumference, step off the chord distances, as $a\,e$, $e\,c$, etc., Fig. 38.

The second method is to draw a line, as $a\,b$, Fig. 38, through the center o of the circle, and then, with a 60° triangle, draw

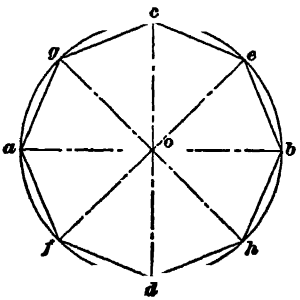


FIG. 37

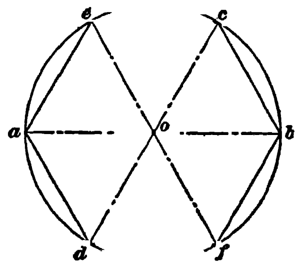


FIG. 38

lines $c\,d$ and $e\,f$. These lines will intersect the circumference and divide it into six equal parts.

53. Regular inscribed polygons are constructed by joining the points of division of an equally divided circle. Thus, Fig. 37 is a regular inscribed octagon, and Fig. 38 is a regular inscribed hexagon. To inscribe a regular polygon of any number of sides in a given circle, it is only necessary to divide the circumference of the circle into the required number of parts and join the points of division.

54. To draw, with triangles, any angle that is the sum or the difference, or a combination of sums and differences, of the angles 90°, 60°, 45°, and 30°.

Certain angles may be laid off directly with the triangles, either singly or in combination. For example, if a line *a c*, Fig. 39, is to be drawn at an angle of 30° with *f b*, it may be laid off with the triangle shown in Fig. 30. If an angle *b a d* of 75° is to be drawn from the point *a* on the line *a b*, it may be laid off by a combination of the triangles shown in Figs. 29 and 30. Similarly, by drawing the line *a e* at an angle of 60° with the angle *b a d*, an angle of 135° will be obtained.

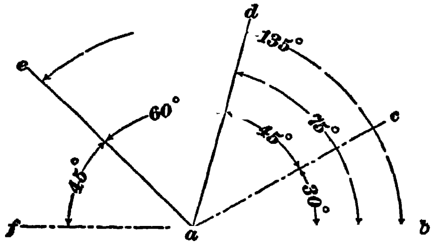


FIG. 39

When the required angle is greater than 90°, it is more easily constructed by laying off an angle that is the difference between the required angle and 180°. Thus, in the preceding example, 45° is the difference between the required angle of 135° and 180°. By laying off the line *a e* at an angle of 45° with *a*, an angle of 135° will be obtained.

PROTRACTOR

55. A **protractor** is a semicircular instrument used for laying off or measuring degrees when triangles cannot be used. It is made of celluloid or metal and is usually graduated to half degrees. The graduations are numbered from each side up to 180, the number of degrees in a half circle, for convenience in laying off degrees on either the right or the left. A protractor with 360 divisions, representing half degrees, is shown in Fig. 40. In laying off angles, the protractor must be placed so that the line *A B* coincides with the line forming one side of the angle and the center *O* is at the vertex of the angle.

For example, let it be required to draw a line making an angle of 54° with the line *O B*, Fig. 41. In this illustration the half-degree divisions are omitted for the purpose of showing the full-degree divisions more clearly. Place the protractor on the

line $O\ B$, with the center at O. With a sharp-pointed pencil make a mark on the paper at the 54° division as indicated

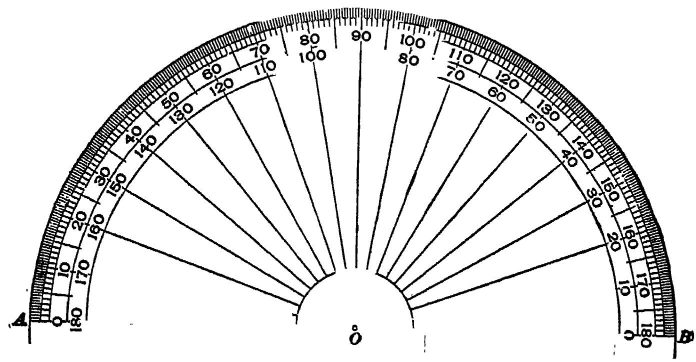


FIG. 40

at C, and draw a line passing through O and C. This line will make an angle of 54° with $O\ B$. Greater exactness will be

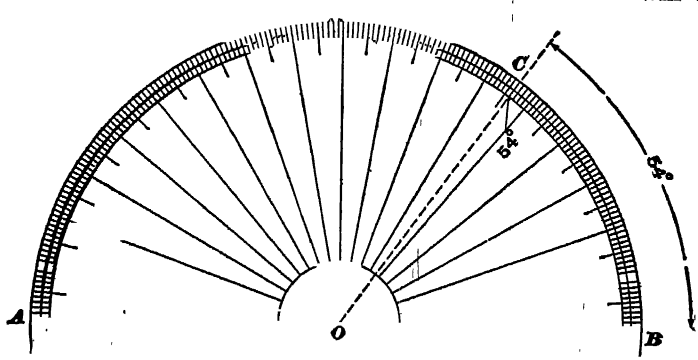


FIG. 41

secured by producing the base line $O\ B$ to the left so that the zero mark on each end of the protractor will rest on it.

COMPASSES

56. The **compasses** are an instrument used for taking or marking measurements, subdividing distances, describing curves, etc. The compasses, next to the T square and the triangles, are used more than any other instrument.

A common form ot compasses is shown in Fig. 42. They consist of two legs joined at the top by means of a fork-shaped brace and held together by means of a screw. The fork carries a head for convenience in handling the instrument and both legs are hinged so that the points may be made to stand in a vertical position, which is necessary in drawing circles. The leg *a* is provided with a socket in which either the pencil point or the pen point *b* shown separately at the right may be inserted. The other leg is also provided with a socket for holding the needle point *c*. In all good instruments, the needle point is adjustable and consists of a round piece of steel of tapering form and having a sharp point. The lengthening bar *d* shown separately is used to extend the leg of the instrument when circles of large radii are to be drawn.

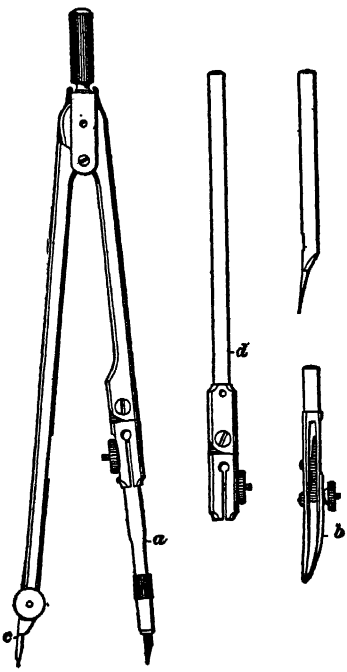


FIG. 42

The point *e* shown at the upper right-hand side of the illustration is sometimes furnished with compasses so that they can be used as dividers. The point is inserted in the leg *a* instead of the pencil point.

57. Use of Compasses.—The following suggestions for handling the compasses should be carefully observed by persons

who are beginning to learn how to use them. A draftsman who handles his instruments awkwardly will create a bad impression, no matter how good a workman he may be. The tendency of beginners is to use both hands for operating the compasses. This should be avoided. The student should learn at the start to open and close them with one hand. To do this the needle-point leg should be held between the third and fourth fingers and the thumb and the other leg held between the first and second fingers, as shown in Fig. 43. With the compasses held in this position, they are inclined sidewise until the side of the hand rests on the paper. This steadies the hand so that the needle point of the instrument may be brought to exactly the desired point. When this has been accomplished the instrument is brought to an erect position, and the leg holding the pencil point is moved to the desired radius by the fingers between which it is held, after which the instrument is held by the head and operated as shown in Fig. 44.

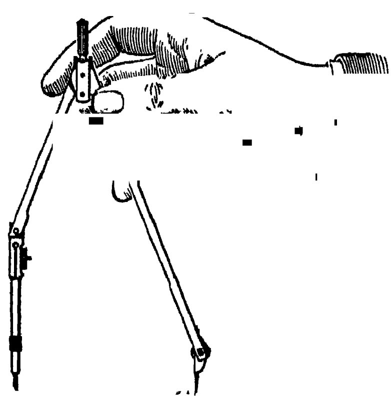


FIG. 43

The compasses must be handled in such a manner that the needle point will not make a large hole in the paper; for this reason use the needle point having a shoulder. In operating the instrument *do not bear on the needle point*, and keep it perpendicular to the paper, as shown in Fig. 44. A slight pressure on the pencil point is necessary. The point of the pencil should be sharpened like the one shown in Fig. 27, but its width should be narrower. *It is important that the pencil point be securely fastened.* If the pencil point is loose it will wabble and the curve will not be made with the same radius at all points. This can be tested by striking arcs of circles in both directions. If the pencil point is properly fixed in position the lines will coincide.

58. Dividers.—Although compasses may be used for laying off distances on a drawing or for dividing lines into equal parts, the dividers can be more conveniently used for these purposes. The common form of dividers is shown in Fig. 45. Dividers are adjusted and then held by the head between the thumb and forefinger the same as the compasses. In stepping off distances, the instrument is turned alternately to the right and the left. If on stepping off equal spaces on a line, it is found that the last space is not of exactly the same length as the first, the instrument must be adjusted, and this procedure must be repeated until the last space is of the same length as those preceding. In making these trial spacings, great care must be exercised not to press the divider points into the paper. If the divider points are pressed into the paper, the spacing cannot be done accurately.

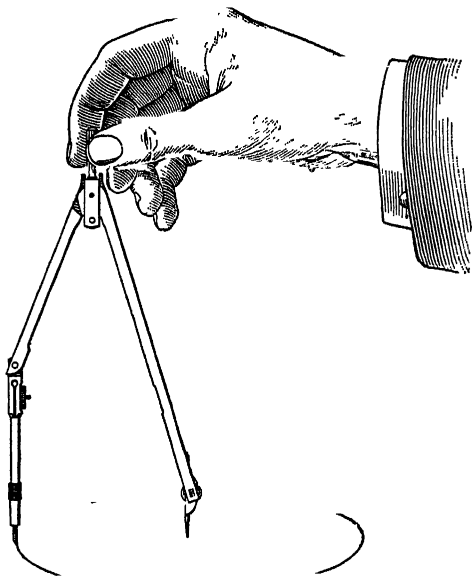


FIG. 44

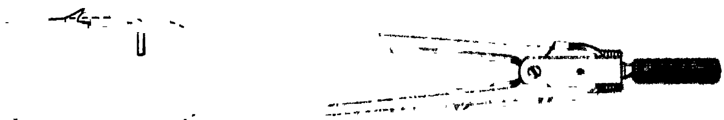


FIG. 45

59. Bow-Pencil and Bow-Pen.—The bow-pencil and bow-pen, shown respectively in Figs. 46 and 47, are small forms of compasses convenient for describing small circles.

Ordinarily the points of the instruments must be adjusted so that both legs are of the same length; but when very small circles are to be drawn, the needle point must be slightly the longer, so that when the point is in the paper the part of the leg above the paper is of the same length as the other leg, otherwise an exact circle cannot be drawn.

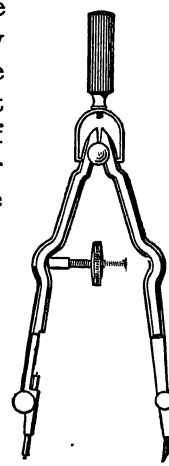


FIG. 46

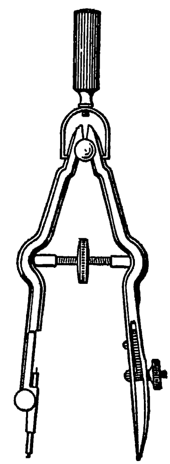


FIG. 47

To draw a circle of a given diameter, the instrument is adjusted approximately to the required radius. It is then placed in an upright position with the point resting on the paper and held by the forefinger pressing lightly on its head. The finer adjustment is then made by turning the adjusting screw with the thumb and the middle finger of the same hand.

60. Use of the Lengthening Bar.—When large circles are to be described, the lengthening bar *d* shown in connection with Fig. 42 must be used. The method of drawing circles by

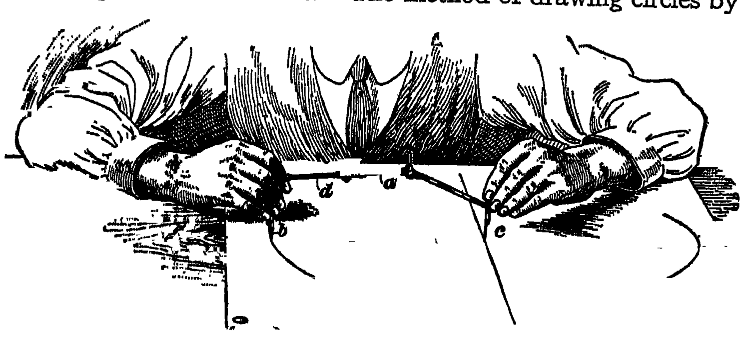


FIG. 48

the use of the lengthening bar is shown in Fig. 48. The lengthening bar is inserted in the socket of the upper part of the leg *a*

and the pen or pencil point is inserted in the socket at the lower end of the lengthening bar. In drawing circles when the lengthening bar is used, the pen point *b* and the needle point *c* are held vertical to the plane of the paper, as shown in the illustration. In such cases the instrument is operated with both hands, as shown.

61. Beam compasses are used for describing circles of very large radii. The instrument consists of a beam *a*, Fig. 49, along which the two pieces *b* and *c* slide; these pieces are provided with clamp screws by which they can be fixed on the beam. The piece *b* carries a needle point, and at *c* either a pencil holder or a pen may be inserted. By fixing *b* and sliding *c* along the beam, the instrument may be set to any radius. Such a compass is not needed in the geometrical drawing work to follow.

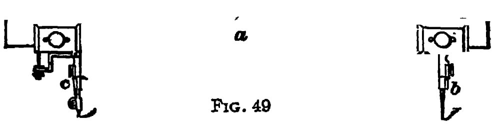


Fig. 49

DRAWING INK AND PENS

62. Drawing Ink.—The ink recommended for drawing work is waterproof liquid India ink. A quill is attached to the cork of every bottle of this ink, by means of which the pen may be filled. The quill is dipped into the ink and the end then passed between the blades of the drawing pen. Not more ink than will fill the blades for a quarter of an inch along their length should be used; if too much is used, the ink is liable to drop and cause blots. The cork should be replaced in the bottle each time the pen is filled.

Before drawing ink is used, the bottle should be well shaken, because some of the ingredients of the ink settle to the bottom, and if the ink is not well mixed the lines will appear gray. If ink becomes too thick it may be diluted by adding a solution composed of 1 ounce of water and 4 drops of aqua ammonia until the ink is of the proper consistency again. Pure water alone should not be used to dilute ink. India ink that has been

frozen cannot be used, as the lines will be very gray and indistinct. The ink bottle should always be kept tightly corked when it is not in use.

India ink dries quickly on a drawing, which is desirable, but it also dries quickly on the blades of the pen, which is not desirable, because it prevents the ink from flowing freely, especially when the pen is adjusted for fine lines. The only remedy is to wipe between the blades frequently with a cloth. The blades should always be wiped out before the pen is laid down for any length of time. If the ink does not flow well, it may be started by moistening the end of the finger and touching it to the point or by drawing a slip of paper between the points of the blades.

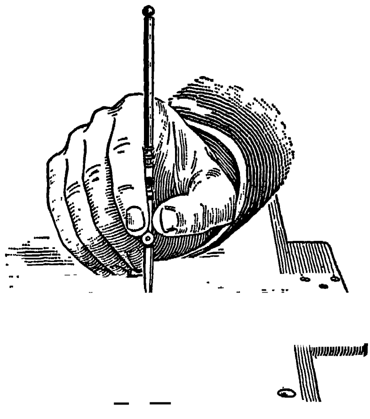
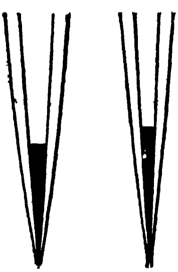


FIG. 50

63. Ruling Pen.—For drawing ink lines other than arcs of circles, the ruling pen shown in Fig. 50 is used. The width of line drawn and the flow of ink are controlled by a small screw, which adjusts the distance between the blades. The proper adjustment of the blades is shown in Fig. 51 (*a*) and the improper adjustment in (*b*). It will be noticed that in (*a*) there is a greater volume of ink at the point of the nibs, which is a good feature, as the ink does not dry as fast as when there is a thin film at the points. If the nibs are brought very close together as shown in (*b*), the points are spread so that they stand apart as shown in the illustration and are, therefore, liable to be injured, the flow of ink is retarded, and ragged, gray lines of irregular thickness will be formed.



(*a*) (*b*)

FIG. 51

When drawing pens become dull, the best plan is to send them to the dealer to be sharpened. The sharpening of pens is highly expert work, and dealers are generally willing to do it at a reasonable price.

64. The ruling pen should be held as nearly perpendicular to the board as possible; the hand should be in the position shown in Figs. 50 and 52 and should press the pen only lightly against the edge of the T square or triangle. If the pen is pressed hard against the edge, the blades will close, with the result that uneven lines will be made. The edge of the T square or triangle should serve simply as a guide for the pen. It will be found that considerable practice is required to make smooth lines. If the pen is held so that only one blade bears on the surface of the paper, the line will almost invariably be ragged on the edge on which the blade does not touch. When the pen is held at right angles to the paper, as shown in Fig. 52, both blades will rest on the paper and if the pen is in good condition smooth lines will result.

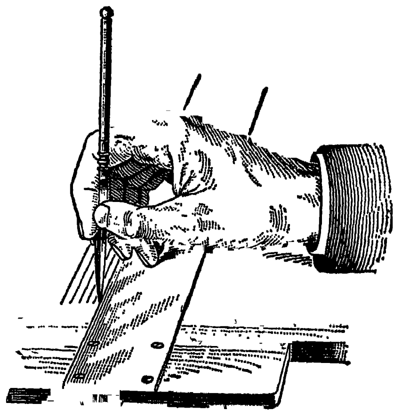


FIG. 52

In some ruling pens a needle point is attached to the head, which can be unscrewed. This needle point is intended for use in pricking holes through a sheet of paper into an underlying sheet on which lines are to be copied.

The ruling pen should always be kept clean. If lint or dust collects on the nibs thick lines or blots will result. Dust that may have accumulated on the drawing paper should be brushed off before lines are drawn.

65. Practice in the Use of Inking-In Instruments. In geometrical drawing great accuracy is required, and only by considerable practice in the use of inking-in instruments

can proficiency be acquired. The beginner should, for drill, draw with ruling pen and ink compasses on good paper lines of different kinds. The practice work should consist of straight lines of different thicknesses and lengths and in different positions, dotted lines (lines made up of a series of short dashes), horizontal and vertical lines that intersect but do not cross, circles and arcs of circles, straight lines tangent to arcs and circles, etc.

CLEANING DRAWINGS

66. A drawing is almost sure to become soiled from the rubbing of the draftsman's sleeves and from dust. This may be prevented to some extent by covering the work, except the part on which work is being done, with paper thoroughly secured at the edges so as not to interfere with the operation of the triangles and T square. One of the most frequent causes of dirt on a drawing is the sliding of the instruments over the

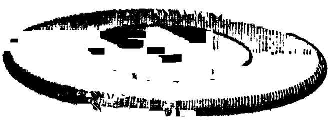


FIG. 53

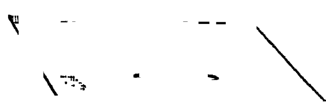


FIG. 54

surface. Celluloid and rubber triangles are especially apt to accumulate dirt, which is transferred to the drawing when these instruments are moved over the paper. It is considered good practice, before commencing a drawing, to clean carefully the scales, triangles, and T square. The drawing board should be dusted before the drawing paper is tacked in place on it, as particles under the drawing paper raise small hills that interfere with the drawing of lines.

After a drawing has been inked in, all soiled spots and pencil lines should be removed with a soft-rubber eraser. This will not injure the ink lines. Before applying the rubber to the drawing, it is a good plan to test it by rubbing it on another sheet of paper to remove any dirt that may adhere to it. If an inked-in line or an ink blot is to be removed, a hard eraser made of a mixture of rubber and emery or glass, such as that

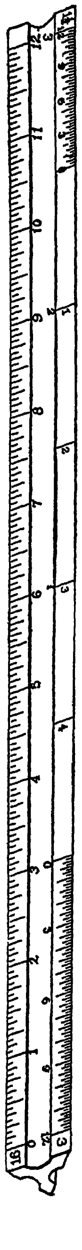


FIG. 55

shown in Fig. 53, should be used. An eraser of this kind cuts away the outer layer of the paper and thus removes the ink lines or blots. It is a good plan, when ink lines are to be removed, to use a celluloid shield such as that shown in Fig. 54. This contains holes of different shapes, so that it is possible to erase particular spots without touching and thus injuring other parts of the drawing. Shields made from thick drawing paper or thin cardboard are also used and have the advantage that slots or holes of the exact shape and size of the spots to be removed may be cut into them; on the other hand, they wear away sooner than those made of celluloid.

SCALES

67. Scales are used for laying off dimensions on drawings. The scales that are usually used are triangular in shape as shown in Fig. 55 or flat with beveled edges. The edges of the triangular scale are graduated for different scales, but as in the geometrical drawing work only the full-size scale is to be used, only this scale will be considered here. The full-size scale is divided, like the ordinary foot rule, into twelve equal parts, called inches, and the inches are subdivided into halves, fourths, eighths, and sixteenths. In using the full-size scale, if it is desired to lay off a dimension as small as a 32d of an inch, it may be done by laying off by eye a point midway between the sixteenth-inch divisions as shown at the left in Fig. 56, in which the subdivisions of the inch are indicated. Dimensions as small as 64ths of an inch may be laid off by eye in the same way by laying off points midway between the 32d-inch divisions. A dimension of $\frac{7}{8}$ inch may be measured by taking seven of the $\frac{1}{8}$-inch divisions, and a dimension of $\frac{5}{16}$ inch may be measured by taking five of the $\frac{1}{16}$-inch divisions, etc.

IRREGULAR CURVE

68. An instrument known as the **irregular curve,** or *French curve,* is used for drawing lines of irregular curvature.

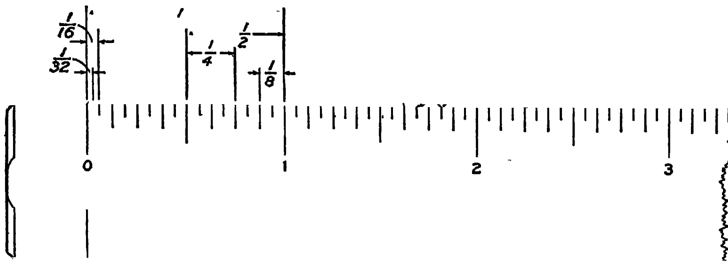


FIG. 56

This instrument is shown in Fig. 57, and its edges have different curvatures at different parts. To draw a line of irregular curvature, the points through which the line is to be drawn must first be determined, and then the curve is placed so that some part of its edge will pass through several of the points determined, then the instrument is shifted as required. Good judgment is required to select the part of the instrument that will give the right curvature and a smooth connection with the part of the line previously made. The curve on the edge of the instrument should be made to pass through at least three of the required points, and if it can be made to pass through a greater number, so much the better, as the fewer the number of parts to be drawn the less liability there will be of getting a line of wavy appearance. Considerable skill is required to

FIG. 57

make curves of irregular shape with a continuous smooth line. It is an advantage to first draw the line in pencil and then ink it in.

To illustrate the use of the irregular curve, let it be required to draw a curved line through the points *a*, *b*, *c*, *d*, etc., Fig. 58. As already said, the curved edge of some part of the instrument should pass through at least three points. By placing the instrument in the first position *A* outlined by dotted lines, the edge is found to pass through five points, *a*, *b*, *c*, *d*, and *e*, and the curved line is drawn through these points, as shown by the full line. To draw the next part of the curve *e f g*, the instrument is shifted to the position *B* and is so adjusted that it will coincide with a part of the curve already drawn and there will be no angle formed where the two parts of the curve join.

In the same way, by shifting the instrument and finding other curves on its edge that will pass through a number of points, the

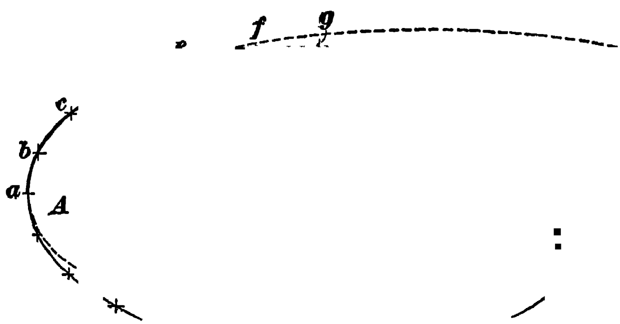


Fig. 58

curved line can be completed. Care should be taken to avoid making sharp angles between connecting sections of the line.

To prevent ink from getting on the edge of the irregular curve and spreading on the drawing, and to make a good line, the blades of the pen must be kept tangent to the edge of the curve; that is, the blade must rest against the edge of the curve tangent to the curved line and must be kept in this position as the pen is moved along the curve. As a precaution against making blots on the drawing, it is advisable to place two thicknesses of paper under the ruler to raise it slightly above the surface of the paper. In this way, if ink should get on the edge of the ruler it will not cause a blot on the drawing.

LETTERING

GENERAL REMARKS

69. On drawings, the headings, explanations, and dimensions are lettered with the pen; script writing is not permissible. Two general styles of lettering are employed on drawings; namely, **single-stroke letters,** which are of two kinds, *slanting* and *vertical*, and **block letters.** Single-stroke letters are adapted for drawings containing many dimensions or on which the space for lettering is limited, as the letters of this form can be condensed without materially affecting their legibility. What is meant by condensing letters is illustrated in Fig. 59, which shows three widths of the same form of letter. The

FIG. 59

slant style is the most natural, as the strokes approximate the direction of the strokes in ordinary writing. Block letters are generally used for main titles and subtitles. It is the usual practice in drawing rooms to adopt a style of letter that is simple in form, legible, and can be made quickly, and all draftsmen in the office are expected to conform to this style. Only the styles of letters in common use, as mentioned preceding, will be discussed here.

For lettering, any good fine-line pen may be used. Gillott's No. 303 pen can be recommended for this purpose. It is possible to make a more uniform line with a pen after it has been used for a short time than when it is new. Waterproof ink dries quickly and for this reason the point should be wiped frequently. A cloth free from lint should be used for this purpose, as the lint would get between the nibs of the pen and clog it.

70. Spacing of Letters.—Next in importance to well-formed letters is the correct spacing of letters in words. By correct spacing is meant the placing of letters at such distances apart as to give the appearance of equal spacing between all letters. The shapes of letters vary, some having slanting sides, some straight sides, and some rounding sides, and others have projecting stems, so that only very general instructions can be given for spacing. Good judgment must be used for this. The letters of a word must be spaced so that the word will have an even appearance and there will be no unduly large white space or dark spot at any point.

More space is required between two letters both of which have straight sides than between two letters one of which has a straight side and one a round side. Less space is required between two letters with rounding sides, as *O O* or *D O*, than is required in either of the preceding cases. The space at the bottom between the two capital letters *A L* should be small so that the space between them at the top will be reduced to the minimum. The capital letters *A W* have parallel sides, consequently considerable space is required between them. The letters that cause the most trouble in spacing are *A*, *W*, *V*, *X*, and *Y*, as, unless good judgment is used, their slanting sides produce unequal white spaces. Letters with projecting strokes, as *F*, *J*, *L*, and *T*, are difficult at times to combine with other letters. The letters that are most easily spaced are those with straight sides, as *H*, *B*, *N*, *D*, etc.

In many drafting rooms a piece of white paper on which horizontal lines are ruled in ink is slipped under the tracing cloth to serve as a guide for lettering. Until one is well trained in lettering, the guide lines should be used; with practice it is possible to make good freehand lettering by using only a base line as a guide.

SINGLE-STROKE LETTERING

SINGLE-STROKE SLANT LETTERING

71. Single-stroke slant letters of the size to be used in this drawing work are illustrated in Fig. 60. The same form of letter is shown on an exaggerated scale in Fig. 61 to indicate more clearly the method of forming the letters and the direction in which the strokes are to be made. The direction of the strokes is indicated by small arrows and the order in which they are made is shown by numerals. To produce well-formed and neat-appearing letters, the direction and order of the strokes as given should be observed.

72. Three elements enter into the construction of this form of letters. These are the straight line, the loop, and the hook, as shown in Fig. 62 (*a*), or modifications of the loop and

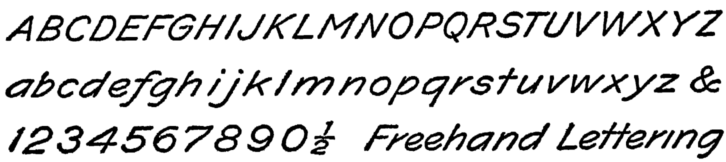


FIG. 60

hook. As will be seen by referring to (*b*), the loop is the main element in the letters *a*, *b*, *d*, *g*, *p*, and *q*, and modifications of it enter into the construction of the *c*, *e*, and *o*. In the letters *a*, *d*, *g*, and *q* the point of the loop is at the top, and in the letters *b* and *p* the point of the loop is reversed. The hook with the turn at the top is used in forming *h*, *m*, and *n*, and the reversed hook is used in *u*. The only difficulty that will be experienced in making the letters *v*, *x*, and *y*, both lower case and capitals, is to draw the sides of the letters at the proper angle. These letters will be well formed if the sides of the *v* and of the *v* part of the *x* and *y* are drawn so that the upper extremities are equally distant from an assumed center line at the angle to which the letters are made, as shown in Fig. 62 (*b*).

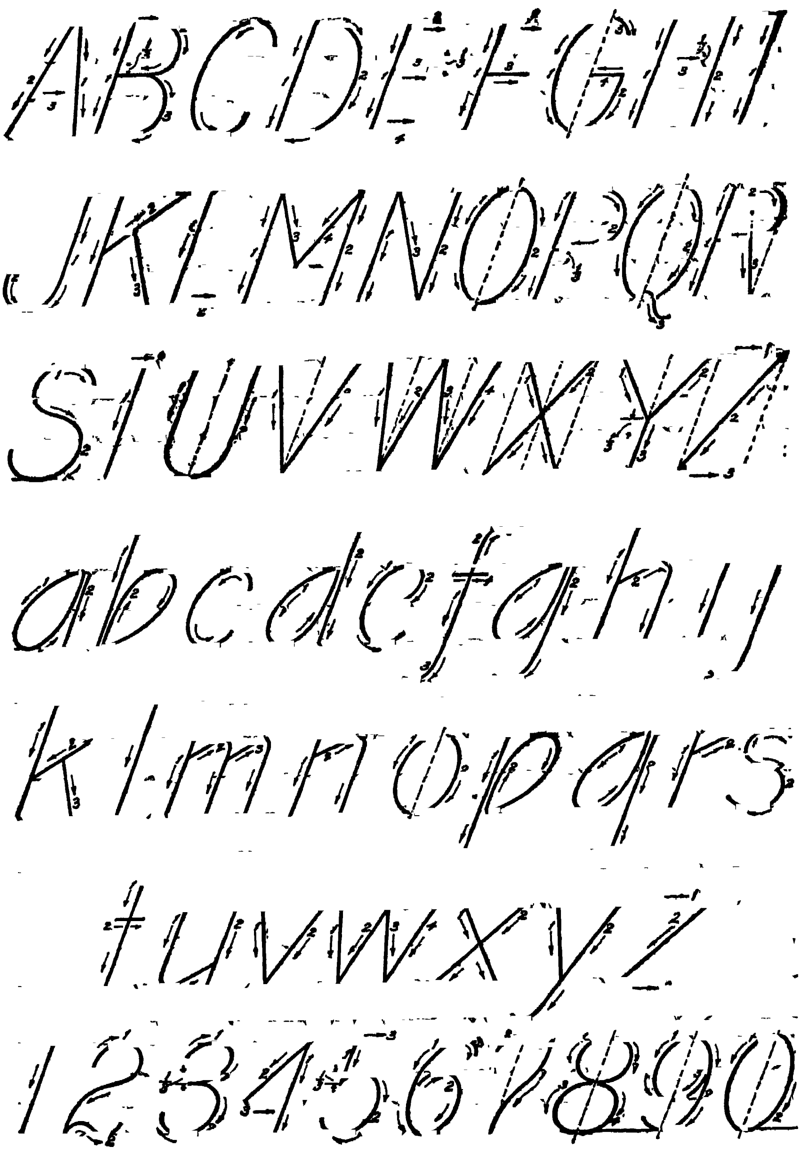


Fig. 61

73. Until the student becomes proficient in drawing the lines of letters at a uniform slant, a templet of the slant used

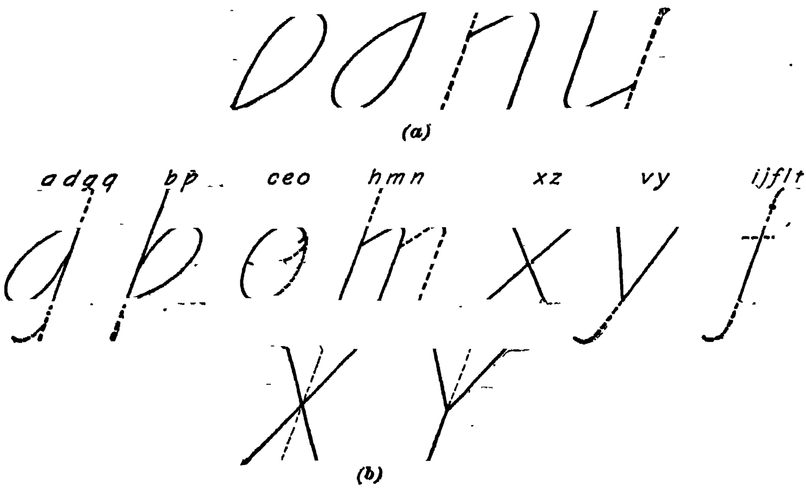


FIG. 62

in the preceding illustrations may be used for drawing guide lines. It may be of either cardboard or wood. The angle, or slant, for the templet may be found by stepping off on a vertical line of any length points to divide it into eight equal parts and then stepping off three of these equal divisions on a horizontal line, as shown in Fig. 63. A line drawn through the extremities of the vertical and horizontal lines will give the hypotenuse of a triangle of the correct angle, or slant, and this angle may be laid off on a templet of any size. If desired, the slant of the letters may be made 60°, and the guide lines for this angle may be drawn by use of the triangle as shown in Fig. 64.

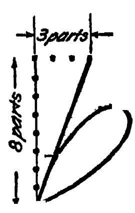


FIG. 63

74. In the drawing work of this Course, the height of the capital letters is to be $\frac{1}{8}$ inch and that of the lower-case letters two-thirds of this, or $\frac{1}{12}$ inch.

The practice in drawing rooms is to letter drawings freehand without the use of guide lines, but until the student becomes proficient in lettering, horizontal guide lines laid off as shown in Fig. 64 may be used for the height of letters. Sheets ruled with these guide lines will be furnished to the student for practice. Proficiency in lettering freehand can be attained only by persistent practice. This may be begun by laying off in pencil slanting guide lines, as shown in Fig. 64, using these until the hand becomes accustomed to the slope and confidence is obtained, after which the practice should be in making the letters without the aid of guide lines.

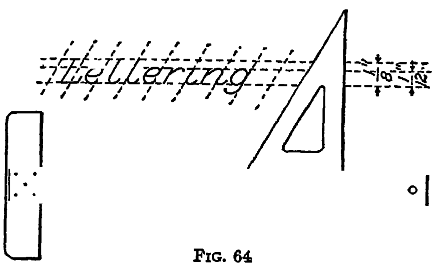


FIG. 64

In practicing the drawing of letters, care should be taken to have the slant and curved lines of all letters extend fully to the guide lines, and the letters should be of proportional width. If these instructions are not carefully followed the letters will present an uneven appearance. The difference in the appearance of a word when the letters are well made and when they are poorly made is shown in Fig. 65. In the first example, the letters are of uniform height and slant and of proportional width, consequently the word presents a good appearance; in the second example, the letters are not uniform in height or slant and are not proportional in width; consequently, the word presents a poor appearance.

Care should be taken to have the ascending stems of letters, as *b*, *d*, *f*, etc., extend fully to the upper guide line; the descending stems of letters, as *f*, *g*, *j*, etc., should extend the same distance below the base guide line that the ascending stems extend above the guide line for the height of the lower-case letters.

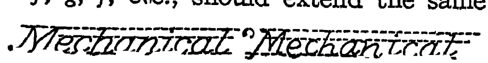


FIG. 65

The rounded letters, as *c*, *e*, and *o*, are usually the most difficult to make, and close attention should be given to their

construction. The loops and curves of letters should be practiced until the hand becomes so accustomed to the movements that they can be made with ease and facility.

75. Another style of slant lettering is shown in Fig. 66. This style differs from the style shown in Fig. 60 only in that

ABCDEFGHIJKLMNOPQRSTUV
WXYZ& 1234567890
abcdefghijklmnopqrstuvwxyz
Cast Iron 1234567890 2'-6¼" dia.

FIG. 66

the letters are made a little rounder and are finished with *spurs*, or *ceriphs*. On account of the ceriphs this style is a little more difficult to make than the style shown in Fig. 60, but with this exception the same instructions apply to both.

76. In lettering drawings, it is common practice to use capital and lower-case letters, in the same manner as in ordinary printed matter. In some drawing rooms, however, capitals and small capitals are used for notes on drawings, in the manner shown in Fig. 67. When so used, the large capitals are made $\frac{1}{8}$ inch high and the small capitals $\frac{1}{12}$ inch.

77. Particular attention should be paid to the formation of numerals, as they are extensively used on drawings, and it is therefore important that they be formed with such exactness, particularly on working drawings, that a mistake in reading them will not be made. The numerals are made $\frac{1}{8}$ inch high, the same as the capital letters. The curve enters into the formation of many of the figures, as will be seen by referring to Fig. 61. The forms of the numerals as there shown should be closely studied and the indicated order and direction of strokes followed. The bottom of the figure *2* and the top of the figure *7* may be either a straight or a curved line.

ALL MATERIAL CAST IRON
UNLESS OTHERWISE ORDERED

FIG. 67

The figures of both the numerator and the denominator of fractions are made the same height as the lower-case letters, as

shown in Fig. 68. By allowing a small space between them for the horizontal dash, the total height of the fraction will be $\frac{7}{32}$ inch. The dividing line of the figures of a fraction should always be a horizontal stroke. A slanting stroke is never used on drawings for this purpose, as it is liable to cause a mistake in reading dimensions.

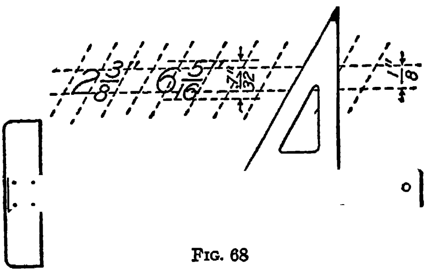


FIG. 68

78. On drawings, the signs for feet and inches are used after numerals. The sign for feet (′) is made with a short, tapering stroke somewhat thicker at the top than at the bottom; the sign for inches (″) is two strokes made in the same way. Care should be exercised to have the strokes begin slightly above the figure and not to make them too long. Careful draftsmen always use a short dash between the figures denoting feet and inches; thus, *10′–5″*. This is done as a precaution against the misreading of dimensions. When only feet are given, this is emphasized by placing after the figures a dash and a cipher followed by the inch sign: thus, *10′–0″*; dimensions less than a foot are indicated thus: *0′–6″*.

SINGLE-STROKE VERTICAL LETTERING

79. A single-stroke vertical letter of the style sometimes used for drawings is shown in Fig. 69. This is a good form of

A B C D E F G H I J K L M N O P
Q R S T U V W X Y Z & a b c d e f
g h i j k l m n o p q r s t u v w x y z
1 2 3 4 5 6 7 8 9 0 $\frac{1}{2}$ Vertical Letter

FIG. 69

letter, but the beginner will find it harder to make the strokes uniform than in the case of slanting letters, which are also more

FIG. 70

easily condensed. The formation of the vertical style of letters is shown to an exaggerated scale in Fig. 70, which also shows the order and direction of the strokes. The forms should be carefully studied and the order and direction of strokes followed. The rules given for the spacing of slanting letters will apply to the spacing of the vertical style.

It will be found necessary on some of the drawing plates of this Course, on account of limited space, to draw the letters less than $\frac{1}{8}$ inch high and more condensed. Guide lines similar to those used in forming the slanting style will be found serviceable.

BLOCK LETTERS

80. The form of letter shown in Fig. 71 is called the **block letter,** and is used for the large headings, or titles, of the Plates.

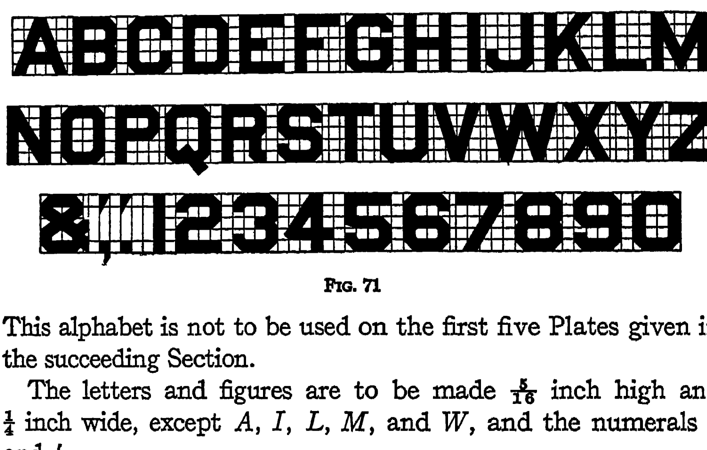


FIG. 71

This alphabet is not to be used on the first five Plates given in the succeeding Section.

The letters and figures are to be made $\frac{5}{16}$ inch high and $\frac{1}{4}$ inch wide, except *A*, *I*, *L*, *M*, and *W*, and the numerals *1* and *4*.

The width of any letter or numeral can be readily determined by referring to the illustration, where each is shown crossed with $\frac{1}{16}$-inch squares.

The width of the spaces between the letters depends on the combination of the letters in words; the best plan to follow in this alphabet is to compare the spacings between the various letters shown in the illustration. Note the space between the

letters *A* and *B* and compare it with that of *B* and *C*, and follow this plan throughout.

To begin to draw the letters, draw in pencil six horizontal lines as guide lines to represent the thickness of the letters at the top, center, and bottom; then by the use of the triangle, the width of the letters, with the proper spaces between them, should be drawn in lead pencil, and the areas of the letters that are to be inked may be penciled over lightly to avoid the possibility of errors in inking. The outlines of the letters should then be inked in with a ruling pen and filled in with a lettering pen. The corners may be drawn with a 45° triangle.

It is well to ink all the vertical lines first, then the horizontal lines, and, finally, the oblique lines.

81. A sloping form of letters somewhat resembling the block letters is shown in Fig. 72. This style of letters will be used for the main titles on subsequent drawing work.

The letters are to be made $\frac{1}{4}$ inch high and $\frac{1}{4}$ inch wide, with the exception of the letters *M*, *W*, and *I*. The width of the

FIG. 72

letter *M* is $\frac{5}{16}$ inch; the letter *W* is $\frac{3}{8}$ inch wide. The letter *I* and the numeral *1* are made with a single stroke, which is $\frac{1}{32}$ of an inch in width.

The slant of the lettering is 60° and is made with a triangle.

To draw these letters, lay off two horizontal guide lines $\frac{1}{4}$ inch apart as shown in Fig. 73.

The rules given for spacing the single-stroke letters apply to this alphabet also.

By using a few simple guide lines, letters may be more easily constructed, as shown in Fig. 73. To draw the letter *A*, use a center line and have the two slanting strokes of the letter

equidistant from it at the base line. The slant sides should not meet in a point at the top, as sufficient allowance must be made for the thickness of the stroke, which is added on the inside when inking.

Similar letters having slanting strokes, as *M*, *V*, and *W*, are drawn in like manner, care being taken that the thickness of the stroke is added on the side where it is required.

The letter *B* is constructed between two slanting guide lines $\frac{1}{4}$ inch apart and the horizontal bar separating the two lobes is drawn slightly above the center. The lobes are connected to the horizontal guide lines at a point determined by a slanting guide line located at a distance of one-third the width of the letter. The upper lobe is also made narrower than the lower one.

The letter *R* is constructed in a similar manner to the letter *B*; the upper part is made the full width and the cross-bar

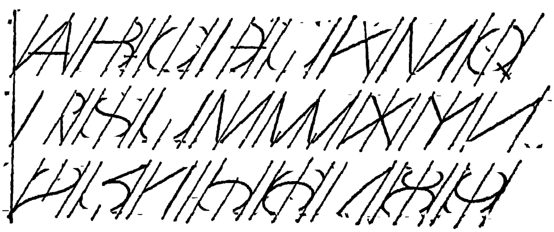


Fig. 73

is midway between the horizontal guide lines. The angle at which the tail is drawn is determined by a guide line located at a distance of one-third the width of the letter from the right-hand edge.

The method of constructing the letters shown in Fig. 73 will be plain from a study of the illustration.

After the letters are penciled they may be inked with a ruling pen; the curves are made freehand with a lettering pen.

The numerals are constructed in like manner to the letters. By taking particular note where the curved parts are tangent to the guide lines, well-formed letters will be produced.

GEOMETRICAL DRAWING

(PART 2)

INTRODUCTION

1. Purpose and Use of Geometrical Drawing Problems.—In this Section are treated the methods of solving certain geometrical problems by the use of drawing instruments.

By the methods here explained it is possible to locate points, lines, angles, etc., without resorting to calculations. The solutions are founded on proved geometrical principles, and the methods given may be used with assurance that correctness in the results depends only on accuracy in drawing. A complete understanding of the methods by which these principles are proved, and upon which the constructions are based, requires a somewhat extended knowledge of geometry, but it is not necessary to understand the proofs in order to make use of the solutions for the purpose for which they are here used and for the every-day work of the draftsman, therefore the proofs are not given.

The problems here treated underlie all work in geometrical drawing, and their application to practical work will be evident as experience is gained. The most important things to be learned in the beginning are (1) to handle the drawing instruments skilfully, (2) to make neat and accurate constructions of the problems, and (3) to print well-proportioned and well-formed title headings and statements of the problems.

When the principles have been mastered, it will not be difficult to do the succeeding plates, which are directly applicable to practical work.

GEOMETRICAL DRAWING PROBLEMS

PRELIMINARY DIRECTIONS

2. The Drawing Plates.—The size of the drawing paper to be used for the drawing work given in this Section is 15 in. ×20 in. Border lines enclosing a rectangle 13 in.×17 in. are to be drawn in pencil and are to be inked in when the drawing is completed, and outside of this border line and ½ inch from it all around another pencil border line is to be drawn. The space on the edges of the sheet outside the pencil lines is to be used for inserting thumbtacks to fasten the sheet on the board and for testing the drawing pen to see whether the ink is flowing well and whether the lines are of the proper thickness. When the drawing is completed the margin of the sheet outside of the pencil lines is to be trimmed off, which will leave the sheet 14 in.×18 in.

3. The drawing work of this Section consists of nine plates, for the making of which full directions are given in the text. The work required on the first five plates consists of solutions of practical geometrical problems that constantly arise in practice, and a knowledge of which is necessary in doing the work on the more advanced plates that follow. A sample copy of the first plate in reduced size will be sent to each student, but no sample copies of the next four plates will be furnished. Sample copies of the sixth and subsequent plates will be furnished in slightly reduced size.

The method of solving each of the problems given for the first five plates should be carefully memorized, so that any one can be instantly applied when the occasion requires without referring to the instructions. Great care should be taken to distribute the different views, parts, etc. on a drawing in such a way that when the drawing is completed one view will not be so near another as to mar the appearance of the work. Until one has gained experience in this, it is advisable to draw

each figure on a separate piece of paper before attempting to locate it on a Plate. This will give familiarity with the construction and also practice in drawing as well as be a help in locating the figure on the Plate. Great care should be taken to lay off dimensions accurately, and the entire drawing should be made in pencil before any part is inked in. In penciling the work, the only distinction that need be made between the construction lines is to make those that are to be inked in a little heavier than those that are not to be. The location of dotted lines may be represented by full pencil lines, which can be made in less time than dotted lines.

The hands should be perfectly clean and should not touch the paper except when absolutely necessary. Construction lines that are to be removed or that are liable to be changed should be drawn lightly, so that the finish of the paper will not be destroyed in erasing them. The methods of removing both pencil and ink lines have been explained in the previous Sections, and those instructions should be carefully followed.

DIRECTIONS FOR SENDING IN WORK

4. The plates are to be sent to the Schools one at a time as they are completed. When you finish Plate I send it to us and then begin work on Plate II. When your first plate is received it will be examined and returned to you promptly with corrections and such suggestions as will aid you in the subsequent work. These corrections and suggestions should receive your careful consideration. In this way you will make better progress than you could otherwise.

It is very important that you comply strictly with the directions. Do not be discouraged if there are a large number of corrections on your early plates; we are merely pointing out ways in which the drawing or lettering can be improved so that your later plates may be as nearly perfect as they can be made. No one can obtain proficiency unless the work is criticized, and we will do our best to help you succeed; we should

not be doing our duty if we did not point out the defects. The *number* of corrections is no indication of our appreciation of the merits of the drawing.

On all plates that you send to us, write your name and address in full in lead pencil on the back of the plates. This should in no case be omitted, as delay in the return of your work will otherwise surely occur.

PLATE I

5. General Directions.—Fasten a sheet of drawing paper to the board as described in the previous Section, then draw the inside border lines to be inked in and outside of these draw the lines to represent the edge of the sheet when it is trimmed, as described in the preliminary directions for this work. The term *drawing* as it will be used hereafter refers to the constructions drawn inside the inked border lines. Before commencing work on Plate I, open the folded sheet inserted at the end of this Section which shows in reduced size Plate I, and spread it before you as a guide. The sample sheet shows the space inside the inked border lines to be divided into six equal rectangular spaces. Now draw midway between the top and the bottom border lines a faint horizontal pencil line on the sheet fastened to the drawing board, thus dividing the space into two equal parts, then divide each of these parts into three equal rectangular spaces by two faint vertical pencil lines. *These division lines are not to be inked in but must be erased when the drawing is completed.*

On the first as well as the next four plates, space for the required lettering must be taken into account. The lettering consists of the word "Problem" and its number and a statement of the problem that is to be drawn in each space. This lettering should not be done before the drawing is finished and inked in, and judgment must be used as to the number of lines the lettering will occupy.

The word *Problem*, as indicated in the sample Plate I, is to be in capital letters; and the statement, *To bisect a straight line*, or any similar note, is to begin with a capital letter. The

lettering may be of the slanting or vertical single-stroke letters explained in the previous Section. If a student is employed where some other style of lettering is used, no objection will be made to that style being used for this drawing work.

The tops of the capital letters in the first line of the statement are placed $\frac{1}{2}$ inch below the upper border line of the space in which the problem is to be drawn. The space between any two lines of lettering is $\frac{1}{8}$ inch, measured from the base line of the first line of lettering to the tops of the capital letters of the second line.

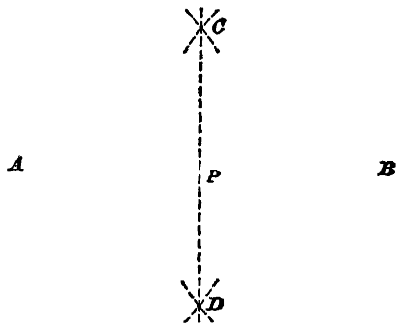


FIG. 1

The height of the capital letters on these plates is to be $\frac{1}{8}$ inch, and that of the small letters is to be two-thirds of this, or $\frac{1}{12}$ inch. To be sure that the heights are uniform, guide lines are drawn in pencil, as indicated in the upper right-hand corner of the sample plate, and the lettering is drawn between them.

The problems are to be centrally located within the six rectangular spaces to insure a neat and well-balanced drawing.

In connection with the descriptions of the plates that follow, the statement of the problem, which is to form the heading, is printed in black-faced type after the problem number.

PROBLEM 1.—**To bisect a straight line.**

See Fig. 1; also Problem 1 of Plate I.

CONSTRUCTION.—With the **T** square as a guide, draw a horizontal line AB $3\frac{1}{2}$ inches long. With one end, as A, as a center and with the compasses set to a radius greater than one-half of the length of the line AB, describe an arc of a circle on each side of the given line; with the other end B as a center and the same radius, describe arcs intersecting the first two in the points C and D. Join C and D by the dotted line CD, and the point P, where it intersects AB, will be the

required point; then, AP equals PB, and P is the middle point of the line AB. As CD is perpendicular to AB, this construction also gives a *perpendicular to a straight line at its middle point.*

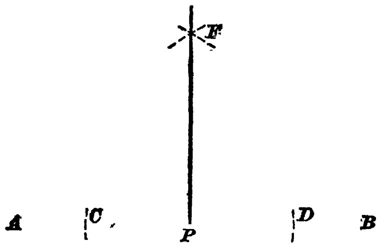


FIG. 2

PROBLEM 2.—**To draw a perpendicular to a straight line from a given point in that line.**

NOTE.—As there are two cases of this problem, requiring two figures on the plate, the line of letters will be run clear across both figures, as shown in sample Plate I.

Case I.—*When the point is at or near the center of the line.* See Fig. 2; also Problem 2, Case I, of Plate I.

CONSTRUCTION.—Draw AB $3\frac{1}{2}$ inches long. Let P be the given point. With P as a center and any radius, as PD, describe two short arcs cutting AB in the points C and D. With C and D as centers and any convenient radius greater than PD, describe two arcs intersecting in E. Draw PE, and it will be perpendicular to AB at the point P.

Case II.—*When the point is near the end of the line.* See Fig. 3; also Problem 2, Case II, of Plate I.

Draw AB $3\frac{1}{2}$ inches long. Take the given point P about $\frac{3}{8}$ inch from the end of the line. With any point O as a center, and a radius OP, describe an arc cutting AB in P and D. Draw DO, and prolong it until it intersects the arc in the point C. A line drawn through C and P will be perpendicular to AB at the point P.

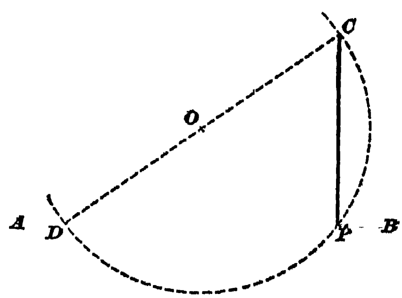


FIG. 3

PROBLEM 3.—**To draw a perpendicular to a straight line from a point without it.**

As in Problem 2, there are two cases.

Case I.—*When the point lies nearly over the center of the line.* See Fig. 4; also Problem 3, Case I, of Plate I.

CONSTRUCTION.—Draw AB $3\frac{1}{2}$ inches long. Let P be the given point. With P as a center and any radius PD greater than the distance from P to AB, describe an arc cutting AB in C and D. With C and D as centers and any convenient radius, describe short arcs intersecting in E. A line drawn through P and E will be perpendicular to AB at F.

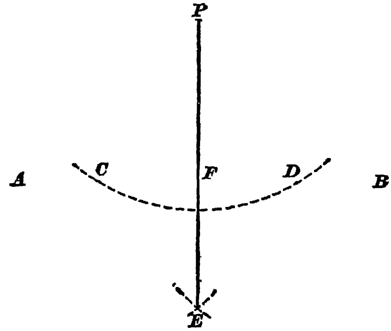


FIG. 4

Case II.—*When the point lies nearly over one end of the line.* See Fig. 5; also Problem 3, Case II, of Plate I.

Draw AB $3\frac{1}{2}$ inches long, and let P be the given point. With any point C on the line AB as a center and with CP as a radius, describe an arc PED cutting AB in E. With E as a center and the distance EP as a radius, describe an arc cutting the arc PED in D. The line joining the points P and D will be perpendicular to AB.

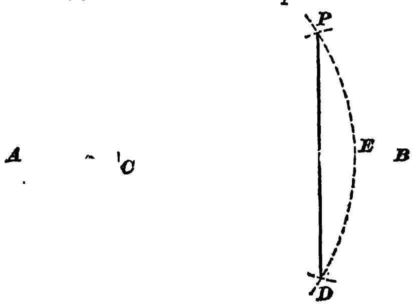


FIG. 5

PROBLEM 4.—**Through a given point, to draw a straight line parallel to a given straight line.**

See Fig. 6; also Problem 4 of Plate I.

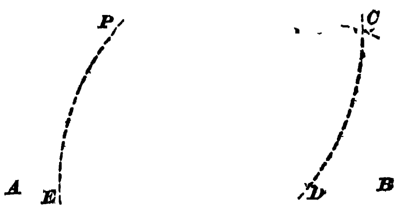


FIG. 6

CONSTRUCTION.—Let P be the given point and AB the given straight line $3\frac{1}{2}$ inches long. With P as a center and any convenient radius, describe

an arc CD intersecting AB in D. With D as a center and the same radius, describe the arc PE. With D as a center and a radius equal to the chord of the arc PE, describe an arc intersecting CD in C. A straight line drawn through P and C will be parallel to AB.

6. These four problems form Plate I. They should be carefully and accurately drawn in with lead-pencil lines and then inked in in the manner described later. It will be noticed that on Plate I, and in Figs. 1 to 6, the given lines are *light*, the required lines *heavy*, and the construction lines, which in a practical working drawing would be left out, are *light dotted*. This system must also be followed in the four plates which are to follow. A single glance enables one to see at once the reason for drawing the figure, and the eye is directed immediately to the required line.

7. In drawing, accuracy and neatness are essential. Be certain that the lines are of *precisely* the length that is specified in the description. When drawing a line through two points, be sure that the line goes through the points; if it does not pass exactly through the points, erase it and draw it over again. If a line is supposed to end at some particular point, make it end there—do not let it extend beyond or fall short. Thus, in Fig. 6, if the line PC does not pass through the points P and C, it is not parallel to AB. By paying careful attention to these points, a great deal of trouble will be avoided.

8. Lines Used on Drawings.—There are five kinds of lines used in drawing, thus:

The light full line.

The dotted line.

The broken-and-dotted line.

The broken line.

The heavy full line.

The **light full line** is used the most; it is used for drawing the outlines of figures and all other parts that can be seen by the eye.

The **dotted line**, consisting of a series of very short dashes, is used in showing the position and shape of that part of the object represented by the drawing which is concealed from the eye in the view shown; for example, a hollow prism closed on all sides. The hollow part cannot be seen; hence its size, shape, and position are represented by dotted lines.

The **broken-and-dotted line**, consisting of a long dash, and with one or two very short dashes repeated regularly, is used to indicate the center lines of the figure or parts of the figure, and also to indicate where a section has been taken when a sectional view is shown. This line is sometimes used for construction lines in geometrical figures.

The **broken line**, consisting of a series of long dashes, is used in putting in the dimensions, and serves to prevent the dimension lines from being mistaken for lines of the drawing.

The **heavy full lines** are made not less than twice as thick as the **light full lines**, and are used for shade lines.

The system according to which shade lines are placed on a drawing will be explained in detail farther on.

9. Inking the Drawing.—To ink the lines of the drawing that has already been made in pencil, first adjust the ruling pen so that the blades practically touch, and put the ink between the blades of the pen with the quill stopper. Then try the pen on the edge of the drawing paper to see whether the blades are set to make a line of the thickness desired and that the ink flows freely. First ink in all the light lines and light dotted lines which have the same thickness; then adjust the pen to the thickness of the heavy lines, test it again, and ink in the lines.

Keep the ruling pen and compass pens clean by wiping the outside and inside of the blades with a damp cloth. India ink dries quickly and soon clogs the blades, so frequent cleaning is necessary to insure an even flow of ink.

10. Lettering and Finishing.—After the drawing has been inked in, it should be lettered. Before attempting this

carefully read the instructions given under the head of Lettering in the previous Section. When the drawing has been finished, the word "Plate" and its number should be lettered at the top of the sheet, outside the border lines, and midway of its length, as shown. The student's name and class letters and number should be lettered in the lower right-hand corner in capital letters. Thus, JOHN SMITH, D Y 618654. The date on which the drawing was completed should be lettered in the lower left-hand corner, also in capital letters. Next erase all pencil lines and clean the drawing by rubbing it very gently with a soft-rubber eraser. Care must be exercised when doing this, or the inked lines will appear of a lighter shade where the eraser has come in contact with them. After the drawing is cleaned, the edges of the sheet should be trimmed off. Finally, write your name and address in full in pencil on the back of the drawing, after which put it in the mailing tube furnished to you and mail it to the Schools according to the directions.

HINTS FOR PLATE I

11. *Do not forget to make a distinction between the thickness of the given and required lines, nor forget to make the construction lines dotted.*

When drawing dotted lines, take pains to have the dots and spaces uniform in length. Make the dots about $\frac{1}{16}$ inch long and the spaces only about one-third the length of the dots.

Try to get the work accurate. The constructions must be accurate, and all lines or figures should be drawn of the length or size previously stated. To this end, work carefully and keep the pencil leads very sharp, so that the lines will be fine.

The lettering on the first few plates, as well as on the succeeding plates, is fully as important as the drawing, and should be done in the neatest possible manner. Drawings sent in for correction with the lettering omitted will be returned for completion.

The reference letters like A, B, C, etc., as shown in Figs. 1 to 6 of the text, are not to be put on the plates

Do not neglect to trim the plates to the required size. Do not punch large holes in the paper with the dividers or compasses.

Before mailing your drawing be sure that it is complete in every detail. Do not attempt to hurry this work. Beginners cannot expect to do what experienced draftsmen have taken some time to acquire. What is worth doing is worth doing well.

PLATE II

12. Draw the pencil border lines and the division lines in the same manner as described for Plate I. The following five problems (5 to 9, inclusive) are to be drawn in regular order, as was done in Plate I, with problems from 1 to 4.

PROBLEM 5.—**To bisect a given angle.***

Case I.—*When the sides intersect within the limits of the drawing.* See Fig. 7.

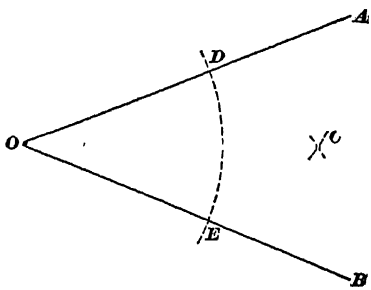


FIG. 7

CONSTRUCTION.—Let AOB be the angle to be bisected. Draw the sides OA and OB $3\frac{1}{2}$ inches long. With the vertex O as a center and any convenient radius, describe an arc DE intersecting OA at D and OB at E. With D and E as centers and a radius greater than half the arc DE, describe two arcs intersecting at C. The line drawn through C and O will bisect the angle; that is, AOC equals COB.

Case II.—*When the sides do not intersect within the limits of the drawing.* See Fig. 8.

CONSTRUCTION.—Draw two lines, AB and CD, each $3\frac{1}{2}$ inches long, and inclined toward each other as shown. With any point E on CD as a center and any convenient radius, describe arc

* Since the statement of this problem is very short, it will be better to place it over each of the two cases separately, instead of running it over the division line, as was done with the long headings of the two cases in Plate I. Put Case I and Case II under the heading, as in the previous plate.

F I G H; with *G* as a center and the same radius, describe arc *H L E F*, intersecting *F I G H* in *H* and *F*. With *L* as a center and the same radius, describe arc *K G J*; with *I* as a center and the same radius, describe arc *J E K*, intersecting *K G J* in *K*

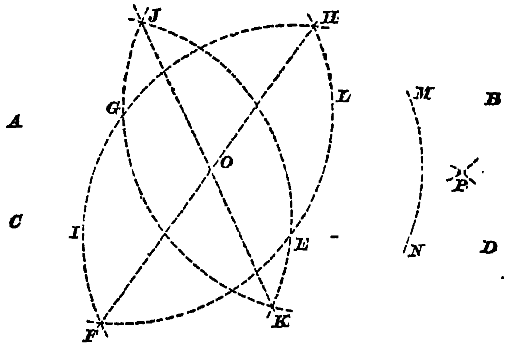


FIG. 8

and *J*. Draw *H F* and *J K*; they intersect at *O*, a point on the bisecting line. With *O* as a center and the same or any convenient radius, describe an arc intersecting *A B* and *C D* in *M* and *N*. With *M* and *N* as centers and any radius greater than one-half *M N*, describe arcs intersecting at *P*. A line drawn through *O* and *P* is the required bisecting line.

PROBLEM 6.—**To divide a given straight line into any required number of equal parts.**

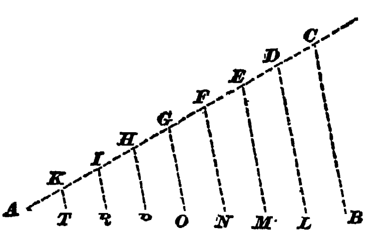


FIG. 9

See Fig. 9.

CONSTRUCTION.—*A B* is the given line $3\frac{7}{16}$ inches long. Suppose that it is required to divide it into eight equal parts. Through one extremity *A* of the line, draw an indefinite straight line *A C*, making any angle with *A B*. Set the dividers to any convenient distance, and space off eight equal divisions on *A C*, as *A K*, *K I*, *I H*, etc. Join *C* and *B* by the line *C B*, and through the points *D*, *E*, *F*, *G*, etc. draw lines *D L*, *E M*, etc. parallel to *C B*, by using the two triangles; these

parallels intersect AB in the points L, M, N, etc., which are equally distant apart. The spaces LM, MN, NO, etc. are each equal to $\frac{1}{8}$ AB. By this method, AB can be divided into any number of parts by spacing on AC the number of parts desired and then drawing the parallels, as explained.

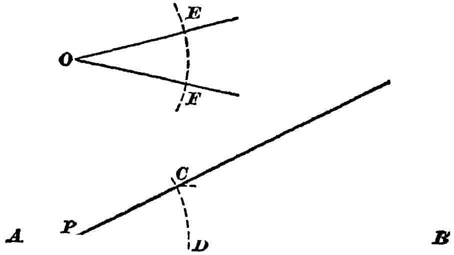


FIG. 10

PROBLEM 7.—**To draw a straight line through any given point on a given straight line to make any required angle with that line.**

CONSTRUCTION.—In Fig. 10, AB is the given line $3\frac{1}{2}$ inches long, P is the given point, and EOF is the given angle. With the vertex O as a center and any convenient radius, describe an arc EF cutting OE and OF in E and F. With P as a center and the same radius, describe an arc CD. With D as a center and a radius equal to the chord of the arc EF, describe an arc cutting CD in C. A line drawn through the points P and C will make an angle with AB equal to the angle O, or CPD equals EOF.

PROBLEM 8.—**To draw an equilateral triangle, one side of which is given.**

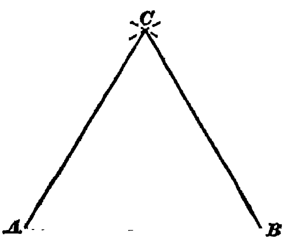


FIG. 11

CONSTRUCTION.—In Fig. 11, AB is the given side $2\frac{1}{2}$ inches long. With AB as a radius and A and B as centers, describe two arcs intersecting in C. Draw CA and CB, and CAB is an equilateral triangle.

PROBLEM 9.—**The altitude of an equilateral triangle being given, to draw the triangle.**

CONSTRUCTION.—In Fig. 12, AB is the altitude $2\frac{1}{4}$ inches long. Through the extremities of AB draw parallel lines CD

and EF perpendicular to AB. With B as a center and any convenient radius, describe the semicircle $CHKD$ intersecting CD in C and D. With C and D as centers and the same radius, describe arcs cutting the semicircle in H and K. Draw BH and BK, and prolong them to meet EF in E and F. BEF is the required equilateral triangle.

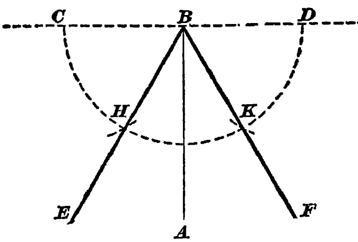


FIG. 12

As each angle of an equilateral triangle is 60°, the same result can be obtained by placing the narrow edge of the 60° triangle along the edge of the **T** square and drawing the line BE to intersect the horizontal lines CD and EF, then reversing the triangle and drawing the line BF to intersect the same horizontal lines.

This problem finishes Plate II. The directions for inking in, lettering, etc. are the same as for Plate I.

PLATE III

13. The border lines for Plate III are to be drawn as explained for the previous plates and the space inside the border lines is to be divided into six spaces in a similar manner.

PROBLEM 10.—**Two sides and the included angle of a triangle being given, to construct the triangle.**

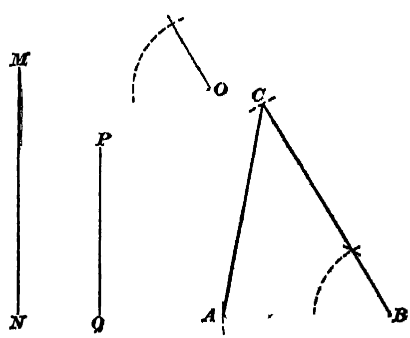


FIG. 13

CONSTRUCTION.—In Fig. 13, make the given sides MN $2\frac{1}{2}$ inches long and PQ $1\frac{7}{8}$ inches long. Let O be the given angle. Draw AB equal in length to PQ. Make the angle CBA equal to the given angle O, and make CB equal in length to the line MN. Draw

CA, and CAB is the required triangle.

PROBLEM 11.—**To draw a parallelogram when the sides and one of the angles are given.**

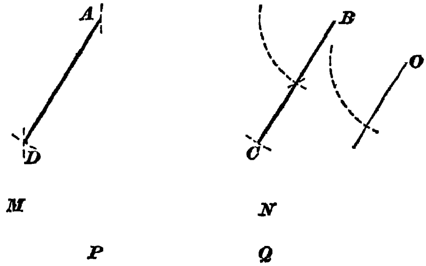


FIG. 14

CONSTRUCTION.—In Fig. 14, make the given sides MN $2\frac{1}{2}$ inches long and PQ $1\frac{7}{8}$ inches long. Let O be the given angle. Draw AB equal to MN, and draw BC, making an angle with AB equal to the given angle O. Make BC equal to PQ. With C as a center and a radius equal to MN, describe an arc at D. With A as a center and a radius equal to PQ, describe an arc intersecting the other arc in D. Draw AD and CD, and $ABCD$ is the required parallelogram.

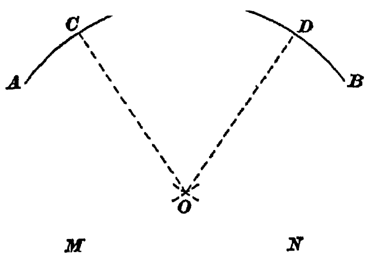


FIG. 15

PROBLEM 12.—**An arc and its radius being given, to find the center.**

CONSTRUCTION.—In Fig. 15, $ACDB$ is the arc, and MN, $1\frac{3}{4}$ inches long, is the radius. With MN as a radius, and any point C in the given arc as a center, describe an arc at O. With any other point D in the given arc as a center and the same radius, describe an arc intersecting the first in O. O is the required center.

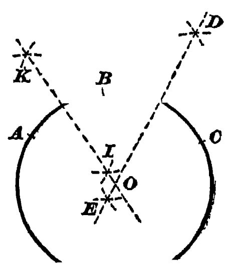


FIG. 16

PROBLEM 13.—**To pass a circumference through any three points not in the same straight line.**

CONSTRUCTION.—In Fig. 16, A, B, and C are the given points. With A and B as centers and any convenient radius, describe

arcs intersecting each other in K and I. With B and C as centers and any convenient radius, describe arcs intersecting each other in D and E. Through I and K and through D and E draw lines intersecting at O. With O as a center and OA as a radius, describe a circle; it will pass through A, B, and C.

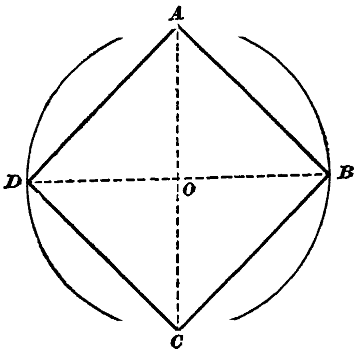


FIG. 17

PROBLEM 14.—To inscribe a square in a given circle.

CONSTRUCTION.—In Fig. 17, the circle $ABCD$ is $3\frac{1}{2}$ inches in diameter. Draw two diameters, AC and DB, at right angles to each other. Draw the lines AB, BC, CD, and DA, joining the points of intersection of these diameters with the circumference of the circle, and they will be the sides of the square.

PROBLEM. 15—To inscribe a regular hexagon in a given circle.

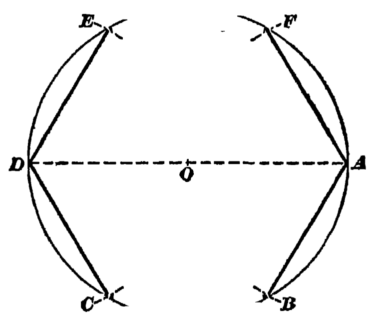


FIG. 18

CONSTRUCTION.—In Fig. 18, from O as a center, with the compasses set to a radius of $1\frac{3}{4}$ inches, describe the circle $ABCDEF$. Draw the diameter DOA, and from the points D and A, with the compasses set equal to the radius of the circle, describe arcs intersecting the circle at E, C, F, and B. Join these points by straight lines, and they will form the sides of the hexagon. This problem completes Plate III.

PLATE IV

14. The first four problems on this plate are more difficult than any on the preceding plates and will require very careful construction. All the sides of each polygon must be of exactly the same length, so that they will space around evenly with the dividers. The figures should not be inked in until the pencil construction is done accurately. The preliminary directions for this plate are the same as for the preceding ones.

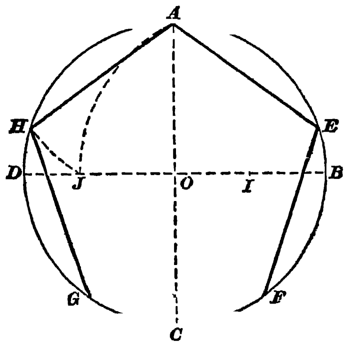


FIG. 19

PROBLEM 16.—**To inscribe a regular pentagon in a given circle.**

CONSTRUCTION.—In Fig. 19, from O as a center, with the compasses set to $1\frac{3}{4}$ inches, describe the circle $ABCD$. Draw the two diameters AC and DB at right angles to each other. Bisect one of the radii, as OB, at I. With I as a center and IA as a radius, describe the arc AJ cutting DO at J. With A as a center and AJ as a radius, describe an arc JH cutting the circumference at H. The chord AH is one side of the pentagon. With the dividers set to this distance, step off the sides AE, EF, etc.

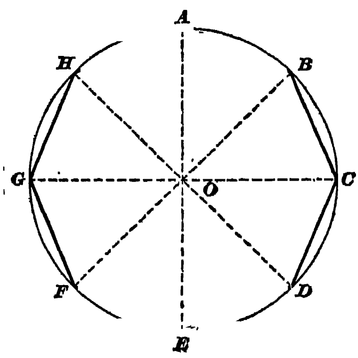


FIG. 20

PROBLEM 17.—**To inscribe a regular octagon in a given circle.**

CONSTRUCTION.—In Fig. 20, from O as a center, with the compasses set to $1\frac{3}{4}$ inches, describe the circle $ABCDEFGH$. Draw the two diameters AE and GC at right angles to each

other. Bisect one of the four equal arcs, as AG, at H, and draw the diameter HOD. Bisect another of the equal arcs, as AC, at B, and draw the diameter BOF. Straight lines drawn from A to B, from B to C, etc. will form the required octagon.

PROBLEM 18.—**To inscribe a regular polygon of any number of sides in a given circle.**

CONSTRUCTION.—With O, Fig. 21, as a center and a radius equal to $1\frac{3}{4}$ inches, describe the circle $A\,7\,B\,C$. Let $A\,7\,B\,C$ be the given circle in which it is required to inscribe a regular polygon of any number of sides. Draw the two diameters $C\,7$ and $A\,B$ perpendicular to each other. Divide the diameter $C\,7$ into as many equal parts as the polygon has sides. We have chosen to inscribe a heptagon, so that the diameter is divided into seven equal parts. Prolong the diameter $A\,B$ and with C or 7 as a center and $C\,7$ as a radius, describe an arc to intersect the vertical center line $A\,B$ at $3'$. Through $3'$ and 2, the second division from C on the diameter $C\,7$, draw the line $3'\,D$ cutting the circumference at D. Draw the chord CD, and it is one side of the required polygon. With the dividers set equal to CD, step off the circumference. The end of the seventh division should coincide exactly with the beginning of the first. The length of each side of any regular polygon is always determined by a line drawn from $3'$ through the second horizontal division on $C\,7$ to intersect the circumference, as at D, in Fig. 21.

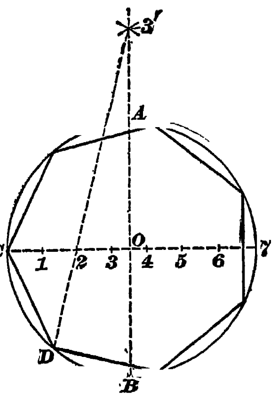


FIG. 21

The draftsman frequently solves this problem by "spacing." This method requires practice, as the spacing may require several adjustments of the dividers, but it should be practiced, as the mechanical draftsman must be expert in the use of drawing instruments.

PROBLEM 19.—**The side of a regular polygon being given, to construct the polygon.**

CONSTRUCTION.—In Fig. 22, let AC be the given side $1\frac{1}{4}$ inches long. The polygon is to have eight sides. Produce AC to B. From C as a center and with a radius equal to CA, describe the semicircle *A 1 2 3 4 5 6 7 B*, and divide it into as many equal parts as there are sides in the required polygon (in this case eight). From the point C, and through the second division from B, as *6*, draw the straight line *C 6*. Bisect the lines AC and *C 6* by perpendiculars intersecting in O. From O as a center and with OC as a radius, describe the circle *C A H G F E D 6*. From C, and through the points *1*, *2*, *3*, *4*, *5* in the semicircle, draw lines CH, CG, CF, etc., meeting the circumference. Joining the points *6* and D, D and E, E and F, etc. by straight lines will complete the required polygon.

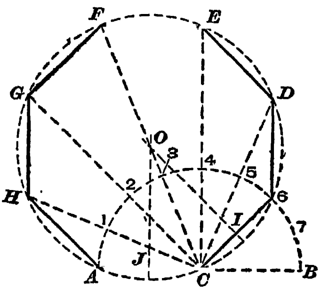


FIG. 22

PROBLEM 20.—**To find an arc of a circle having a known radius, which shall be equal in length to a given straight line.**

NOTE.—There is no exact method, but the following approximate method is close enough for all practical purposes when the required arc does not exceed one-sixth of the circumference.

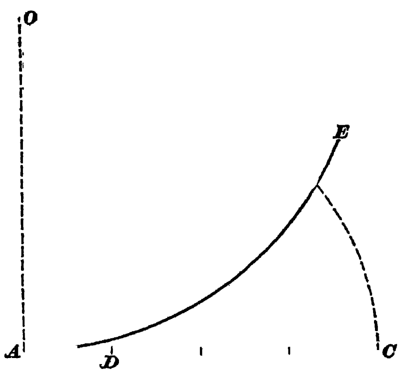


FIG. 23

CONSTRUCTION.—In Fig. 23, let AC be the given line $3\frac{1}{2}$ inches long. At A, erect the perpendicular AO, and make it equal in length to the given radius, say 4 inches long. With OA as a radius and O as a center, describe the arc ABE. Divide AC into four equal parts, AD being the first of these

parts, counting from A. With D as a center and a radius DC, describe the arc CB intersecting ABE in B. The length of the arc AB very nearly equals the length of the straight line AC.

PROBLEM 21.—**An arc of a circle being given, to find a straight line of the same length.**

This is also an approximate method, but close enough for practical purposes, when the arc does not exceed one-sixth of the circumference.

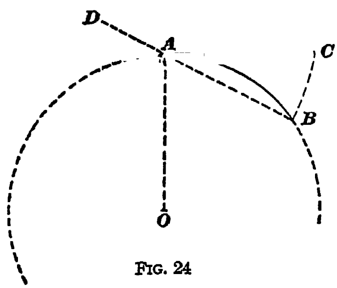


FIG. 24

CONSTRUCTION.—In Fig. 24, let AB be the given arc drawn with the radius OA. For this problem, choose the arc so that the radius will not exceed $1\frac{3}{4}$ inches. At A, draw AC perpendicular to the radius (and, of course, tangent to the arc). Draw the chord AB, and prolong it to D, so that AD equals $\frac{1}{2}$ the chord AB. With D as a center and a radius DB, describe the arc BC cutting AC in C. AC will be very nearly equal to the arc AB.

PLATE V

15. On Plate V there are five problems instead of six, as on the preceding plates. It should be divided into six equal parts, or divisions, as in the previous cases. The two right-hand end divisions are used to draw in the last figure of Plate V, which is too large to put in one division.

PROBLEM 22.—**To draw an egg-shaped oval.**

CONSTRUCTION.—In Fig. 25, on the diameter AB, which is $2\frac{3}{4}$ inches long, describe a circle $ACBG$. Through the center O, draw OC perpendicular to AB, cutting the circumference $ACBG$ in C. Draw the straight lines BCF and ACE. With B and A as centers and the diameter AB as a radius,

describe arcs terminating in D and H, the points of intersection with BF and AE. With C as a center, and CD as a radius, describe the arc DH. The curve $ADHBG$ is the required oval.

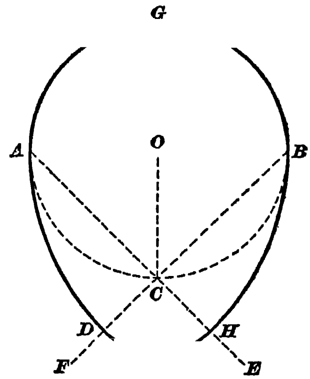


FIG. 25

PROBLEM 23.—To draw an ellipse, the diameters being given.

CONSTRUCTION.—The ellipse to be constructed is to have a long diameter, or major axis, of $3\frac{1}{2}$ inches and a small diameter, or minor axis, of $2\frac{1}{4}$ inches. Draw two concentric circles, using radii of lengths equal to one-half those of the given diameters of the ellipse. Then, as in Fig. 26, through the common center O, draw vertical and horizontal center lines, AC being equal to the long diameter and BD equal to the short diameter of the ellipse.

From the center O to the circumference of the large circle, draw any desired number of radial lines, as r, s, t, u, v, which also pass through the circumference of the smaller concentric circle in points r', s', t', u', v'.

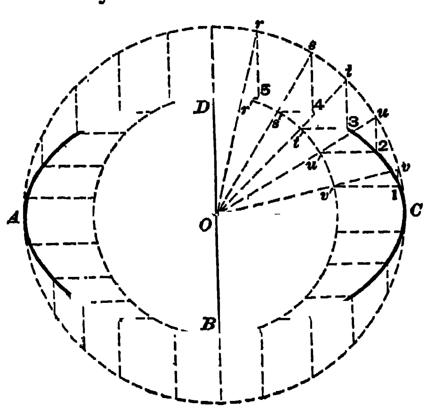


FIG. 26

From the points where the radial lines intersect the circumference of the larger circle, as r, s, t, u, v, draw vertical lines parallel to BD, and from the points where the radial lines intersect the circumference of the smaller circle, as r', s', t', u', v', draw horizontal lines parallel to AC; the points where these lines intersect, as *5*, *4*, *3*, *2*, *1*, are points through which the curve of the ellipse is to be drawn.

Trace a curve with the pencil through the points thus found by placing a celluloid irregular curve on the drawing in such a manner that one of its bounding lines will pass through three or more points, judging with the eye whether the curve so traced bulges out too much or is too flat. Then adjust the curve again, so that its bounding line will pass through the next three or more points, and so on, until the curve is completed.

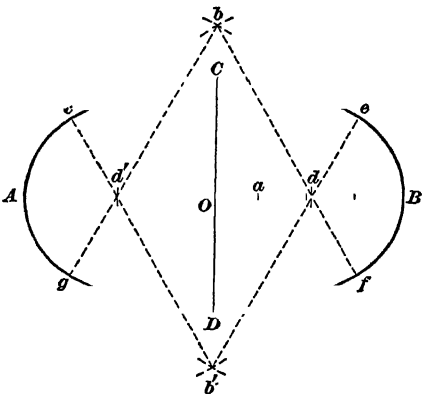


Fig. 27

Considerable practice is required to be able to draw a good curved line in this manner, and the general appearance of the curve thus drawn depends a great deal on judgment and accuracy.

When inking, do not fail to place a piece of paper folded double the thickness under the irregular curve to raise it slightly from the paper to guard against making blots caused by the ink flowing onto the under side of the curve.

Problem 24.—**To draw an ellipse by circular arcs.**

An ellipse made with circular arcs is not true in form but the method is very convenient for many purposes.

Construction.—See Fig. 27. Use the same dimensions as in Problem 23. On the major axis $A\,B$, set off $A\,a$ equal to $C\,D$, the minor axis, and divide $a\,B$ into three equal parts. With O as a center, and a radius equal to the length of two of these parts, describe arcs cutting $A\,B$ in d and d'. On $d\,d'$ as a side, construct two equilateral triangles $d\,b\,d'$ and $d\,b'\,d'$. With b as a center and a radius equal to $b\,D$, describe the arc $g\,D\,f$ intersecting $b\,d\,f$ and $b\,d'\,g$ in f and g. With the same radius and b' as a center, describe the arc $c\,C\,e$ intersecting $b'\,d'\,c$ and $b'\,d\,e$ in c and e. With A and B as centers and a radius $A\,c$ or $B\,e$, describe arcs cutting $A\,B$ very near to d' and d.

and with g and l as centers, and a radius equal to $g\,C$, draw the arcs $j\,C\,h$ and $m\,D\,i$.

PROBLEM 25.—**To draw a parabola, the axis and longest double ordinate being given.**

EXPLANATION.—The curve shown in Fig. 28 is a **parabola.** This curve and the ellipse are the bounding lines of certain sections of a cone. The line $O\,A$, which bisects the area included between the curve and the line $B\,C$, is called the **axis.** Any line, $B\,A$ or $A\,C$, drawn perpendicular to $O\,A$, and whose length is included between $O\,A$ and the curve, is

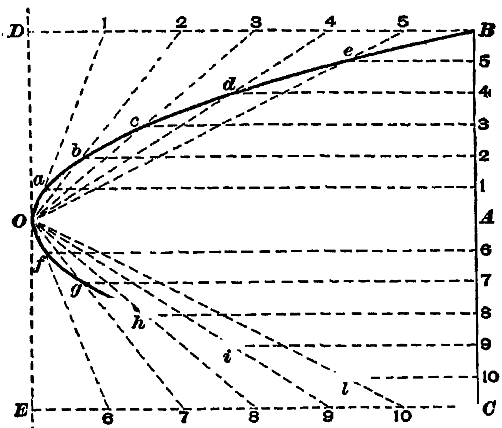


FIG. 28

called an **ordinate.** Any line, as $B\,C$, drawn from one side of the curve to the other, is called a **double ordinate.** The point O is called the **vertex.**

CONSTRUCTION.—Make the axis $O\,A$ equal to $3\frac{1}{2}$ inches, and the longest double ordinate $B\,C$ equal to 3 inches. $B\,A$, of course, equals $A\,C$. Draw $D\,E$ through the other extremity of the axis and perpendicular to it; also draw $B\,D$ and $C\,E$ parallel to $O\,A$ and intersecting $D\,E$ in D and E. Divide $D\,B$ and $A\,B$ into the same number of equal parts, as shown (in this case six); through the vertex O, draw $O\,1$, $O\,2$, etc. to the points of division on $D\,B$, and through the corresponding

points *1*, *2*, etc., on *A B*, draw lines parallel to the axis. The points of intersection of these lines, *a*, *b*, *c*, etc., are points on the curve, through which it may be traced. In a similar manner, draw the lower half *O f g h i l C* of the curve.

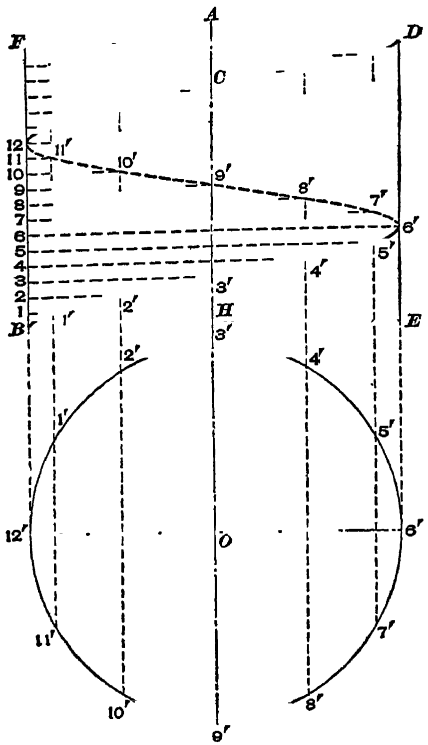


FIG. 29

PROBLEM 26.—To draw a helix, the lead and the diameter being given.

In Fig. 29 is shown a rectangle *F D E B*, which represents a cylinder standing in a vertical position, as indicated by the axis *A O* passing through its center. Below this view of the cylinder is a circle that represents the bottom view. On the cylindrical surface *F D E B* is shown a curved line known as a **helix.** As this helix advances around the cylinder it describes a curved path *B 1′ 2′ 3′*, etc., as shown. The distance that the curved line advances lengthwise of the cylinder during one complete revolution is the **lead.** The term *pitch* is sometimes used instead. The use of the term pitch in this connection is likely to cause confusion, as will be seen from the discussion of screw threads when the subject of *Mechanical Drawing* is reached. The lead of the helical curve is the distance from a point on the curve to a corresponding point on the same curved line measured parallel to the axis of the cylinder when the curve has made one complete revolution on the cylinder.

The helical curve and the lead can be illustrated more clearly by cutting a piece of paper A to triangular shape, as shown in Fig. 30. This triangular piece of paper, or pattern, has a right angle at a; its length $a\,b$ is equal to one and one-half times the circumference of the cylinder, its side $a\,c$ is equal to the height of the helical curve, and its hypotenuse is equal to one and one-half turns of the curve. If this piece of paper is wound on the cylinder B, its horizontal length $g\,d'$ will be found to cover one-half the cylinder; its horizontal length $n\,e'$ will cover the entire surface, its full length will cover the cylinder one and one-half times, and its hypotenuse will produce on the cylinder the helical curve $c'\,d\,e\,b'$. The distance $c'\,e$ is the lead.

Construction.—Only Fig. 29 is to be drawn on the plate and, as mentioned before, this figure is to occupy two spaces. The

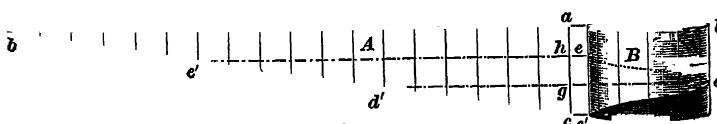


Fig. 30

diameter of the cylinder is 3½ inches, the lead is 2 inches, and a turn and a half of the helix is to be shown. The rectangle $F\,B\,E\,D$ is a side view of the cylinder, and the circle $1'\,2'\,3'\,4'$, etc. is a bottom view. It will be noticed that one-half of a turn of the helix is shown dotted; this is because that part of it is on the side of the cylinder that cannot be seen. Lines that are hidden are drawn dotted. Draw the axis $O\,A$ in the center of the space. Draw $F\,D$ 3½ inches long and 4 inches from the top border line; on it construct a rectangle whose height $F\,B$ equals 3 inches. Take the center O of the circle 2¾ inches below the point H on the axis $A\,O$, and describe a circle having a diameter of 3½ inches, equal to the diameter of the cylinder. Lay off the lead from B to 12 equal to 2 inches, and divide it into a convenient number of equal parts (in this case 12), and divide the circle into the same number of equal parts, beginning at one extremity of the diameter $12'$–O–$6'$, drawn parallel to $B\,E$. At the point $1'$ on the circle divisions.

erect *1′–1′* perpendicular to $B\,E$; through the point *1* of the lead divisions, draw *1–1′* parallel to $B\,E$ to intersect the perpendicular in *1′*, which is a point on the helix. Through the point *2′*, erect a perpendicular *2′–2′*, intersecting *2–2′* in *2′*, which is another point on the helix. So proceed until the point *6′* is reached; from this point to the point *12*, the curve will be dotted. It will be noticed that the points of division *7′ 8′*, *9′*, *10′*, and *11′* on the circle are directly opposite the points *5′*, *4′*, *3′*, *2′*, and *1′*; hence, it was not necessary to draw the lower half of the circle, since the point *5′* could have been the starting point, and the operation could have been conducted backwards to find the points on the dotted upper half of the helix. The other curved full line of the helix can be drawn in exactly the same manner as the first half.

REPRESENTATION OF OBJECTS

16. An object as it appears to the eye may be represented in a drawing by an outline such as would be derived by tracing the form of the object from a photograph, and such an outline would be a good example of a perspective drawing. The perspective drawing shown in Fig. 31 gives as clear an idea of an object as a view of the object itself would give. If, however, the edges *a b* and *c d* are measured on the drawing they will not be found to be the same, as they would be if measured on the object itself. Similarly, the edges *e f*, *e b*, and *b c*, which are equal on the object itself, will not be found to be the same when measured on the drawing.

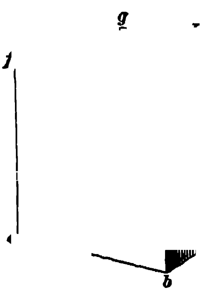


FIG. 31

A perspective drawing, therefore, is unsuitable for obtaining measurements of different parts of an object. The true lengths of lines in perspective can be found only with great difficulty even by persons who are perfectly familiar with the method. What is known as a **projection drawing** is the kind of drawing

universally made when an object is to be represented in its true dimensions, and this kind of a drawing can be made more easily than a perspective drawing.

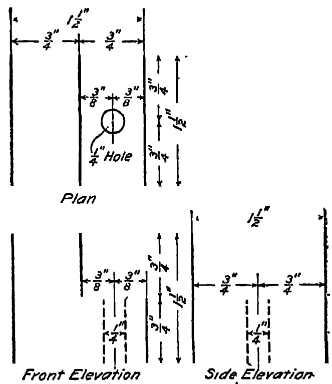


FIG. 32

In general, the size and shape of an object may be shown in three views, namely, a *plan view*, a *front elevation*, and a *side elevation*, as shown in Fig. 32. These views would be meaningless if one did not understand what the lines meant, and to understand what they mean it is necessary to understand how they are obtained.

In Fig. 33, the object shown in Fig. 31 is assumed to be placed within a glass case, and the various views take their names from the different positions of the observer in his view of the object through the transparent sides, or planes. The top plane *1–2–6–7*

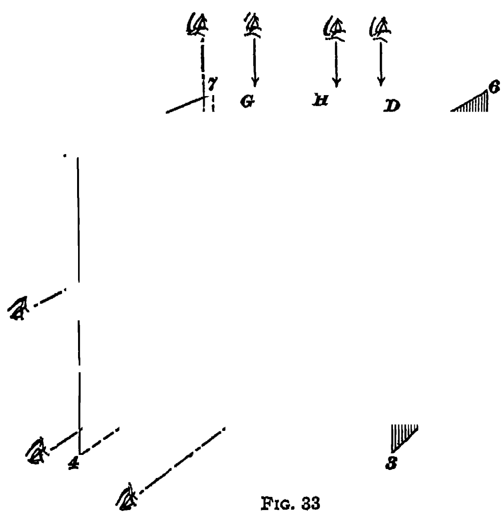


FIG. 33

is parallel to the top surfaces *a d k j* and *f g h i* of the object; the front plane *1–2–3–4* is perpendicular to the top plane and

parallel to the surface *a b e f i j* of the object. The side plane *2–3–5–6* is at right angles to the top plane *1–2–6–7* and also to the front plane *1–2–3–4*; and the planes *2–3–5–6* and *1–2–3–4* are parallel, respectively, to the side and the front surfaces of the object.

17. To understand how views are obtained on any of these planes, assume that we are to obtain first the top view, or plan.

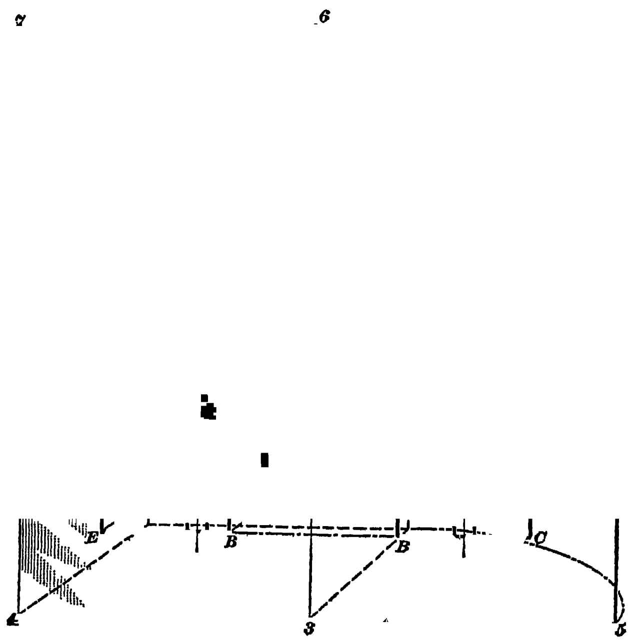


FIG. 34

Imagine that the eye in Fig. 33 is brought successively vertically over the points *a, d, h, g, f,* and *i* on the object in the imaginary glass case and that straight lines pass from these points through the transparent horizontal plane to the eye. These lines are sometimes referred to as *lines of sight* but more commonly as *projection lines*, which are always at right angles to the plane. If, now, the top plane is regarded as a sheet of paper and the points where the assumed projection lines intersect

this plane, as A, D, H, G, F, I, are indicated on the paper and the points are connected by straight lines, we will have the rectangle shown in the top view, or plan, of Fig. 32. The round hole in the surface $a\ d\ j\ k$ is represented on the plan by a circle. Before the circle is described its center is located by the intersection of two lines drawn at right angles.

The front elevation $F\ I\ J\ A\ B\ E$, Fig. 33, in the front plane *1–2–3–4* and the side elevation $H\ D\ C\ B\ A\ I$ in the side plane *2–3–5–6* are obtained in a similar manner to the top plane by assuming the line of sight to be in a horizontal direction. In the front and side views, the round hole, instead of being represented by a circle as in the top view, is represented by two dotted lines with a center line between them.

In practice, the three views of an object, instead of being shown on three different planes, are represented on one plane as on a sheet of paper. The relation of the views on a single plane will be better understood by assuming that the top plane of Fig. 33 is hinged along the edge *1–2* and that the side plane is hinged along the edge *2–3*, so that these planes can be swung into the plane of the front view, as shown in Fig. 34. The illustration then represents exactly what is shown in Fig. 32.

18. The lines on which the planes are hinged represent the axis between the views, and they intersect in a center from which arcs may be described to transfer points from one view to another, as shown in Fig. 35. The dimensions may, however, be transferred by the use of dividers, and as this is the customary practice, the lines of the planes and of the axis of revolution are omitted from the drawing. In many cases it is immaterial which view of a drawing is made first; in some cases it is the most convenient to draw the plan first and in other cases to draw the elevation first.

In the preceding discussion, only three views have been considered. The relative location of these views, however, must be fixed firmly in mind in order to read a mechanical drawing easily and to obtain a correct idea of the form of the objects drawn. In all drawings made by this method, the *plan* is always shown at the top, the *front elevation* below it,

and the *right side elevation* at the right of the front elevation, or plan. The views on the three other planes may be obtained in a similar manner by projecting points from the object. The *bottom plan* would be projected on the plane below the front elevation; the *left side view* would be projected to the left of

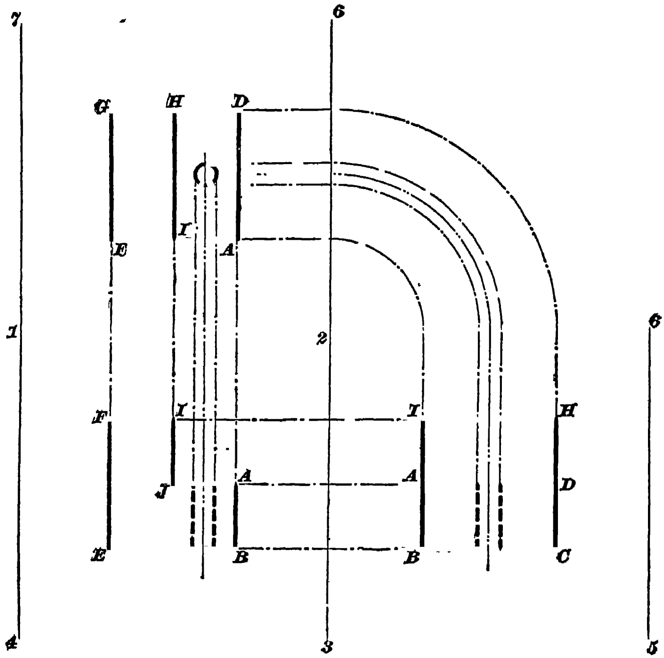


Fig. 35

the front elevation; and the *rear view* would be projected either to the right or the left of the side elevations.

19. The foregoing explanation of the manner in which the different views of an object are obtained on a single plane should have prepared the student for the work on the drawing plates that are to follow, and the work on these plates may now be commenced. In order to distinguish clearly the figure numbers on the plates from those of the figures given in the text, the figure numbers referring to the plates are printed in heavy-faced type.

DRAWING PLATE, TITLE: PROJECTIONS—I

20. Arrangement of Figures.—On the preceding plates, the space was divided into a number of equal parts so that the figures of the different problems could be centrally located. The sizes and shapes of the figures for the plates that are to follow differ so widely that it is not advisable to attempt to divide the space into equal parts. Instead, the location of each figure will be given, so that the drawing will have a neat appearance. In no case should a figure come nearer than $\frac{3}{4}$ of an inch to the border line. For this, as well as succeeding plates, border lines are first to be drawn as explained for Plate I.

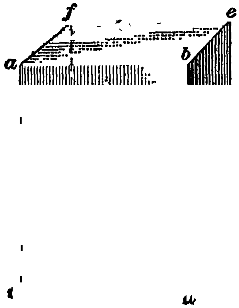


FIG. 36

21. Views of a Rectangular Prism.—In Fig. 36 is shown a rectangular prism standing on one of its ends and of which three views are to be drawn, as shown in Fig. 1 of the drawing plate. The reason why the three views occupy the relative positions shown will be clear from Fig. 37, which shows the prism X standing in a glass case and the different views projected on the planes. The front elevation $ABDC$ on the plane *1–2–3–4* is a projection of the face $abdc$ of the prism, the side elevation $BEHD$ on the plane *2–6–5–3* is a projection of the face $b\,e\,h\,d$, and the top view $ABEF$ on the plane *1–2–6–7* is a projection of the face $abef$. The views of Fig. 37 are lettered to correspond with

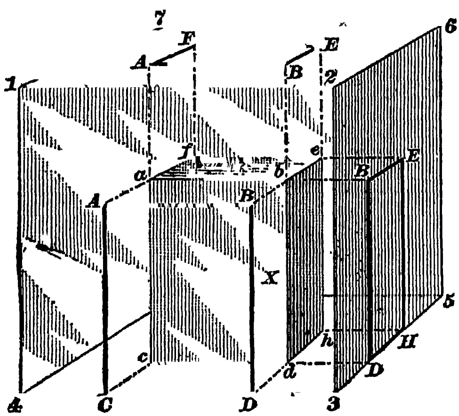


FIG. 37

those on the drawing plate, and if the planes are revolved in the manner previously explained the views will occupy the position shown on the plate.

The prism is 2 inches long, $1\frac{1}{2}$ inches wide, and $\frac{3}{4}$ inch thick. To construct the figure, draw a pencil line across the entire sheet $4\frac{3}{4}$ inches from the upper border line as a base line for the first four figures, and on this base line, $\frac{3}{4}$ inch from the left-hand border line, locate the point C of the front elevation. From the point C lay off $C\,D$ on the base line the width of the prism, $1\frac{1}{2}$ inches. From these points draw two vertical lines of indefinite length and on these lines lay off the distance $C\,A$ and $D\,B$ equal to the height of the prism, 2 inches. The front elevation is completed by drawing the horizontal line $A\,B$.

To construct the plan, draw horizontal lines $A\,B$ and $F\,E$ $\frac{3}{4}$ inch apart to connect the extended vertical lines of the front elevation. To construct the side elevation, extend the line $A\,B$ and from the points D and H on the base line draw two vertical lines $\frac{3}{4}$ inch apart to intersect the extended horizontal line in the points B and E. This completes the side elevation.

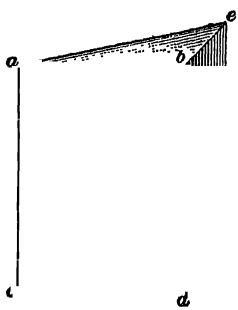


Fig. 38

22. Views of Wedge, Triangular in Plan.—In Fig. 38 is shown a perspective view of a triangular prism, or wedge, of which a drawing is to be made. This triangular prism may be considered as one-half of the prism shown in Fig. 36, since, if the prism were cut vertically along a line from a to e, two triangular prisms like that shown in Fig. 38 would be produced.

To make the drawing, construct the front elevation first, locating point C 5 inches from the left-hand border line, then the side elevation, and the plan last. The front and side elevations are constructed in exactly the same way as for Fig. 1. The plan, however, is triangular in shape, and a line from A to E completes this view.

23. Views of Wedge, Triangular in Front View. If the rectangular prism of Fig. 36 were cut through from front

to back along a diagonal line drawn from *a* to *d*, a wedge like that shown in Fig. 39 would be produced. The front elevation of this wedge will be a triangle. To obtain the front elevation shown in Fig. **3**, first locate point *C* $7\frac{5}{8}$ inches from the right border line and erect two vertical lines from *C* and *D* at a distance of $1\frac{1}{2}$ inches from each other. Make *A C* 2 inches in height, and from the point *A* in the elevation draw a line to *D*. Then *A C D* is the desired elevation. It will be observed that the triangle *A D C* is just half of the rectangle *a b d c* forming the front face of the rectangular prism of Fig. **1**. The plan and side elevation of the wedge in Fig. **3** are like those of the rectangular prism and are drawn in the same way.

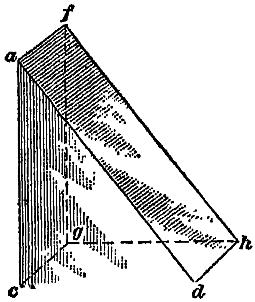


FIG. 39

24. Views of Wedge, Triangular in Side View.—In Fig. 40 is shown a perspective view of a form of wedge obtained by cutting the rectangular prism shown in Fig. 36 along a diagonal line from *b* to *h*. The plan and front elevation in Fig. **4** correspond with those of the prism and are drawn in the same manner. The side elevation, however, is triangular in form and may be drawn by extending the lines *A B* and *C D* horizontally; between the lines thus prolonged draw a vertical line *B D*. From *D* set off horizontally a distance of $\frac{3}{4}$ inch, locating *H*, and draw *B H*. The side elevation *B D H* is thus a triangle, and is one-half of the rectangle *D B H E* forming the side elevation of the rectangular prism shown in Fig. **1**.

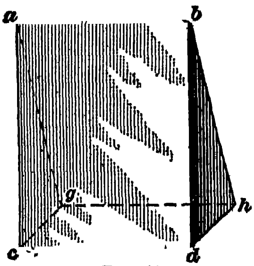


FIG. 40

25. Necessity for Three Views of Some Objects.—A comparison of Figs. **2**, **3**, and **4** with Fig. **1** will serve to show why three views of some objects are necessary in order to fully define their shapes. In Figs. **1** and **2**, the side elevations and front elevations are alike and if no other views were given

it would be impossible to determine whether the objects represented by the two drawings were the same or not. However, as soon as the plan of each is added, it is seen immediately that one is a rectangular prism and the other a triangular prism. In Figs. 1 and 3, the plans and side elevations are of the same size and shape, but the front elevations differ, thereby showing the difference between the two objects illustrated in the drawings. Again, the front elevations and plans of Figs. 1 and 4 are alike and no difference in the objects represented could be determined by these views alone. By adding the side elevation of each, the difference is at once clearly shown.

26. Views of a Cylinder.—In Fig. 5 is shown a plan and an elevation of a cylinder, 2 inches long and $1\frac{1}{4}$ inches in diameter. The plan, or top view, of this object is a circle, and the front elevation is a rectangle. No side elevation is given, for the reason that it is not needed, as it is the same as the front elevation.

Draw the plan view of Fig. 5 first. Begin by drawing the center line $p\,q$ $6\frac{1}{4}$ inches below the upper border line, and this line may be extended to serve as a guide line for the plan views of Figs. 6 and 7. Erect the vertical center line $m\,n$ $1\frac{3}{8}$ inches from the left-hand border line, and with the point of intersection of this line with the line $p\,q$ as a center, describe a circle with a radius of $\frac{5}{8}$ inch, thus completing the plan. To construct the elevation, draw a horizontal base line $3\frac{1}{8}$ inches below $p\,q$, and extend this line across the plate to serve as a base line for Figs. 6 and 7. Project two vertical lines from the horizontal diameter of the circle to intersect the horizontal base line in the elevation. On the center line $m\,n$ locate a point 2 inches above the base line and through this point draw a horizontal line extending to the projected sides, thus completing the elevation.

27. Views of a Hexagonal Prism.—In Fig. 41 is shown a perspective view of a regular hexagonal prism, three views of which are to be drawn as shown in Fig. 6. The prism is 2 inches long and $1\frac{1}{4}$ inches thick, measured between any two parallel sides.

The plan, which is a regular hexagon, is drawn first, with two of the parallel sides horizontal. To begin, draw a center line *m n* $5\frac{3}{4}$ inches from the left-hand border line and at right angles to it draw the center line *p q*. From the point of intersection *O* as a center and a radius equal to one-half of the distance between two parallel sides ($\frac{1}{2} \times 1\frac{1}{4}'' = \frac{5}{8}''$), describe a circle. Now, use the **T** square to draw two horizontal lines of indefinite length through the points of intersection of this circle with the center line *m n*. By means of the **T** square and 60° triangle, draw *A B* and *C D* through *O*, which makes the angles *A O q* and *C O p* each equal 60°. This is done by keeping the longer of the two short sides of the triangle vertical, and passing the pencil along the hypotenuse.

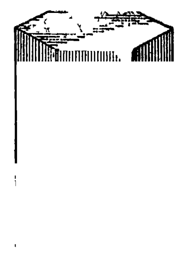


Fig. 41

From the points of intersection of *A B* and *C D* with the two horizontal lines, draw *E K* and *H G* parallel to *C D*. From *F* and *I*, the points of intersection of *C D* with the same two horizontal lines, draw *F G* and *K I* parallel to *A B*. This completes the hexagon, or plan, of the prism.

To draw the front elevation of the prism, measure off from the point *L* on the center line *m n* a distance of 2 inches, locating the point *J*, and through this point draw a horizontal line for the top edge of the prism. Project the points *K*, *I*, *H*, and *G* of the plan to *K′ G′*, as shown by the projection lines; and through the points of intersection of these projection lines with *K′ G′* draw the vertical lines *K′ M*, *I′ N* *H′ P*, and *G′ Q*, thus completing the front elevation.

For convenience in this case, the side elevation of the prism is drawn at the left of the front elevation. It could just as well have been drawn at the right, as in previous cases. To draw the side elevation, extend the line *K′ G′* to the left and draw the center line *t v*. Lay off on each side of the center line a distance equal to one-half the distance between the parallel sides, or $\frac{5}{8}$ inch, and draw vertical lines *E′ X*, *I″ N′*. Draw the vertical line *M′ K″*, which will coincide with the axis, and the side elevation is complete. The sides of the hexagon in the plan should be accurately drawn before projecting them.

28. Views of a Hexagonal Pyramid.—In Fig. 42 is shown a perspective view of a hexagonal pyramid for which views are to be drawn as shown in Fig. 7. The base is a regular hexagon of the same size as the base of the prism in Fig. 6. The plan, therefore, is constructed in the same way as for the prism. This gives A, B, C, D, E, F, Fig. 7. From O, which is located $6\frac{13}{16}$ inches from the right-hand border line and at the intersection of the two center lines, draw lines $O\,A$, $O\,B$, $O\,C$, $O\,D$, $O\,E$, and $O\,F$, which are the horizontal projections of the slanting edges of the pyramid. Then, to draw the front elevation, lay off $O\,I$ on the center line $m\,n$ equal to the altitude, and through I draw the line $A'D'$. Project the points D, E, etc. of the plan on $A'D'$, as shown by the projection lines, and join them with the point O by the lines $A'O$, $F'O$, $E'O$, and $D'O$; these lines are the vertical projections of the edges of the pyramid. The side elevation can be easily drawn, and does not require a special description. The length of the base $B\,F$ is equal to the distance between the parallel sides, or $1\frac{1}{4}$ inches.

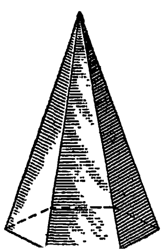


Fig. 42

29. Views of a Rivet.—In Fig. 8 is shown a plan and an elevation of a rivet $\frac{7}{8}$ inch in diameter and having a button head $1\frac{1}{2}$ inches in diameter. A side elevation is not necessary, as it is exactly the same as the front elevation. Draw the plan view first, locating the center line $p\,q$ at a distance of $6\frac{9}{16}$ inches from the lower border line. At the distance of $4\frac{1}{8}$ inches from the right-hand border line draw the vertical center line $m\,n$. With the bow-pencil set to a radius of $\frac{3}{4}$ inch, describe a circle from the point O for the plan view and also a dotted circle from the same center with the bow-pencil set to a radius of $\frac{7}{16}$ inch, or one-half the diameter of the rivet. Now draw a horizontal line of indefinite length at a distance of $1\frac{3}{4}$ inches below $p\,q$ for the base line $A\,B$ of the head of the rivet. Project lines from the points on the circumference of the large circle where it intersects the center line $p\,q$ to intersect the line locating the base line $A\,B$ of the head. On the center line $m\,n$ lay off

the point O $\frac{17}{32}$ inch above the base line $A B$ for the height of the head, and from a center located on the line $m n$ describe with the compasses an arc passing through the points A, O, and B. Now, to complete the figure, project the diameter of the rivet from the dotted circle in the plan and draw the lines $E G$ and $F H$. The irregular line $G H$ indicates that only a part of the rivet is shown. This is done so that too much space will not be taken up on the drawing. This part is shown sectioned to represent wrought iron.

30. Views of a Square-Headed Bolt.—In Fig. 43 is shown a perspective view of an ordinary square-headed bolt of which a plan and elevation is to be drawn as shown in Fig. **9.** The bolt is $\frac{7}{8}$ inch in diameter, has a head $1\frac{3}{8}$ inches square, and is $\frac{13}{16}$ inch thick. To construct the figure, draw a vertical center line $m n$ at a distance of $1\frac{7}{16}$ inches from the right-hand border line and another center line $A B$ at a distance of $6\frac{5}{8}$ inches from the lower border line. Construct the plan view of the head by making a square $1\frac{3}{8}'' \times 1\frac{3}{8}''$. A quick method of constructing the square is to describe a circle with a radius of $\frac{11}{16}$ inch, or one-half the dimensions of the square, and draw vertical and horizontal lines tangent at the points where the center lines $A B$ and $m n$ cut the circle. At a distance of 2 inches below the center line $A B$ draw a horizontal line of indefinite length to locate the line $C D$ representing the lower edge of the head in the elevation. Project the vertical side lines of the plan to the elevation, thus defining the width of the head in the points $C D$. Measure off from the line $C D$ vertically a distance of $\frac{13}{16}$ inch, for the height of the bolt head. The head is to be chamfered at the corners, and this is defined by an arc drawn tangent to the top edge of the bolt head and with a radius of $1\frac{3}{8}$ inches from a point on the vertical center line $m n$. A dotted circle is now drawn in the plan view with a radius of $\frac{7}{16}$ inch, or one-half of the diameter of the bolt. From the plan view the diameter of the bolt is projected to

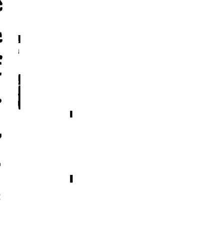


FIG. 43

the elevation and lines $E G$ and $F H$ are drawn. The circle in the plan view is shown dotted to indicate that the bolt cannot be seen in that view.

31. Views of a Distance Piece.—Fig. **10** shows a distance piece used to separate two machine parts a certain distance, and for which a drawing is required. To construct the figure, draw a horizontal center line $n m$ $1\frac{3}{4}$ inches from the lower border line, extend it across the entire sheet to serve for Figs. **11** and **12**, and through this line at a distance of $6\frac{1}{4}$ inches from the left border line draw a center line $p q$ at right angles to it. Draw the side elevation first by describing a circle with a 1-inch radius for the two end flanges. Then describe a dotted circle with the bow-pencil set to a radius of $\frac{5}{8}$ inch to represent the outside diameter of the cylinder and inside it a circle to represent a $\frac{3}{4}$-inch hole through it. Now, draw two vertical lines 4 inches apart to define the total length of the distance piece, and next draw parallel lines $\frac{1}{2}$ inch from each end, to indicate the thickness of the flanges. Project from the side elevation to the front elevation horizontal lines to define the diameter of the flanges and the cylinder between the flanges, and also dotted lines to define the hole through the piece. Use a radius of $\frac{1}{8}$ inch for the fillets at A, B, C, and D and round the corners at E, F, G, and H with the same radius.

32. Views of a Circular Cast-Iron Washer.—In Fig. **11** is shown a circular cast-iron washer square in cross-section. In this case, instead of making an elevation and plan, only an elevation is drawn and a sectional view is taken through this elevation along the line $p q$; that is, the washer is imagined to be cut along the line $p q$, with the part to the left removed, which gives a sectional view as shown.

Begin by drawing a vertical center line $p q$ at a distance of $6\frac{1}{4}$ inches from the right-hand border line and, with the compasses set to a radius of 1 inch, describe a circle representing the outside diameter of the ring, and from the same center, and with a radius of $\frac{1}{2}$ inch, describe another circle representing the inside diameter. Now, to the left of this view, lay off two vertical lines $\frac{1}{2}$ inch apart to represent the thickness of the

washer, and from the elevation project horizontal lines to the sectional view. The corners of this section should be rounded with a radius of $\frac{1}{16}$ inch.

33. Section Lines.—In order to distinguish a sectional drawing without any possibility of mistake, so-called section lines are employed. These are usually made by laying a 45° triangle against the edge of the T square and drawing a series of parallel lines as equally spaced as can be judged by the eye. For cast iron, these lines are full, thin lines, all of the same thickness, and must not be drawn too near together. The method of sectioning for other materials will be given later. It is not customary to draw the section lines in pencil, but to wait until the outlines of the drawing have been inked in and then make the section lines directly with the drawing pen. Section lines should be spaced not less than $\frac{1}{16}$ inch apart, unless, as in Figs. 11 and 12, the drawing is of such small dimensions as to cause a sectioning of this spacing to look coarse. In these two figures, space the section lines a full $\frac{1}{32}$ inch apart. The only parts of the figure to be sectioned are those surfaces that are produced by cutting the ring along the line $p\,q$, both views of the figure being projections on the vertical planes.

34. Views of a Cast-Iron Cylindrical Ring.—Fig. 12 is a cast-iron cylindrical ring. It is shown in elevation and in sectional elevation. The dimensions given suffice for the drawing of the figure without further explanations. The extremities of the diameter of the small circle in the elevation are projected to the sectional side elevation in the points A and B.

35. Inking In of Figures.—The pencil lines showing the construction of the figures of the drawing plate Projections—I are now complete and the entire plate is to be inked in, including the dimension lines and figures. When inking in a drawing, it is generally best to draw the circles and curved lines first and the straight lines last. This enables the draftsman to blend the straight lines into the curved lines so that their points of meeting cannot be detected. Also, tangent lines can be

drawn with better success, and the time for inking in is shortened. It will be noticed that on the drawing plate some of the straight lines are heavy and some light, and that parts of full-line circles are heavy and other parts light. The heavy lines are shade lines, which are described later in this Section. The dotted lines used to indicate the parts of the figures that are hidden must be of the same thickness as the full lines; the construction lines and center lines should be very thin.

36. Dimension figures are to be made $\frac{1}{8}$ inch high and the fractions are to be made $\frac{7}{32}$ inch high. On some of the plates that are to follow there may not be room for figures of this size. In such cases, the figures may be made smaller, but care must be taken to make them *clear*. Until the student has had sufficient practice in lettering, he should draw guide lines in pencil for the dimension figures, as shown in Fig. 44. All the figures should have the same slant of 60°, and, when printing fractional dimensions, the *whole* fraction should have the same slant as the figures; that is, the denominator should not be vertically under the numerator but a little to the left, so that a slanting guide line would pass through the middle of both the numerator and denominator, as shown in the illustration. The dividing line between the numerator and denominator of a fraction is always to be a horizontal line. A slanting division line is never permissible on drawings.

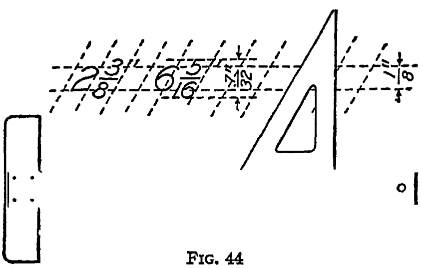


FIG. 44

37. Dimension and extension lines must be light, broken lines of the same thickness as the center and construction lines. Care should be exercised to make the arrowheads as neatly as possible and of a uniform size. They are made with a Gillott's No. 303 pen, or any other fine lettering pen, and their points must touch the extension lines, as illustrated in Fig. 45. Do not make arrowheads too flaring.

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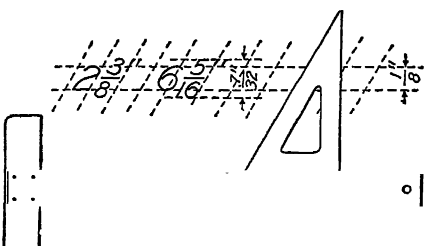

FIG. 44

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When putting in the dimensions, care should be taken to give *all* that would be needed to make the piece which the drawing represents, but usually the same dimensions would not be given on different views. On complicated drawings it is sometimes advisable to duplicate dimensions, as this facilitates the reading of the drawing. In Fig. 1 of the plate, the length is given in the front elevation as 2 inches, and it is obviously unnecessary to give the same dimension in the side elevation. Again, the dimension lines should be put where they would be most likely to be looked for. In Fig. 10 the diameter of the central part of the distance piece is marked 1¼ inches in the front elevation; it could have been marked on the side elevation, as the diameter of the dotted circle, but a person wishing to find the size of this part would naturally look for it in the front elevation. This is also true of the diameter of the flange. The diameter of the hole could be on the side elevation or front elevation, but it is put on the side elevation because it is denoted there by a full line, while in the front elevation the hole is shown by dotted lines. The lines of a drawing should never be crossed by dimension lines when possible to avoid it. In Figs. 2 and 4 the ¾-inch and in Fig. 3 the 2-inch dimension lines have been placed outside of the figures, thus avoiding crossing the bounding lines of the figures.

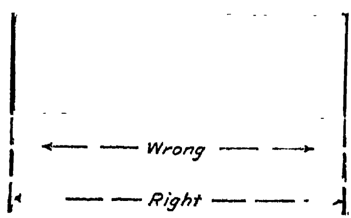

FIG. 45

All the figures used for dimensions shown on this and succeeding plates should be inked in on the drawing, but the letters used to describe the different objects should be omitted. The title should be made in block letters as shown on the sample plate. The date, name, class letters and number are to be put on as in the preceding plates.

DRAWING PLATE, TITLE: PROJECTIONS—II

38. When the surfaces of objects are parallel to the planes on which the views are drawn, the projected views represent the object in its true dimensions. When, however, a machine whose parts are placed at different angles is to be drawn, the projected outlines of the surfaces will not be represented in their true dimensions but will be foreshortened. The figures on the plate Projections—II represent objects similar to those in the preceding plate, but in some instances the surfaces are inclined to the planes of projection, consequently they are not represented in their true dimensions; that is, they appear foreshortened.

39. Rectangular Prism Parallel to the Horizontal Plane but Inclined to All Vertical Planes.—On the preceding plate, Projections—I, the front elevations *A B C D* of Figs. **1**, **2**, and **4** represented outlines true to their dimensions, as the front face *a b c d*, Fig. 37, of the prism in each case was parallel to the front vertical plane.

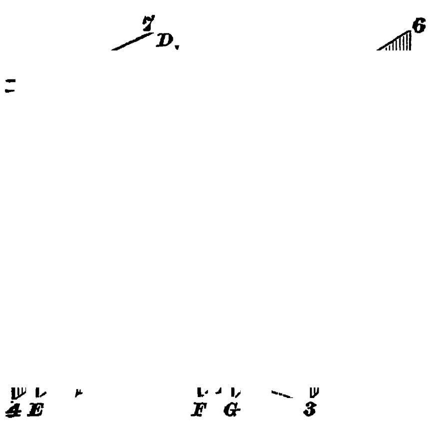

FIG. 46

In Fig. **1** of the plate Projections—II is shown the drawing of a prism whose front and side faces are not parallel to the front and side planes, and consequently are not represented in their true dimensions. The drawing will be better understood by referring to Fig. 46, which shows a prism *X* whose face *a b f e* makes an angle of 30° with the front vertical plane, and consequently its projection on this plane is a foreshortened view. The face *a b c d* of the prism, however, is parallel to the plane *1–2–6–7*, therefore its projection *A B C D*, or plan view, will be in its

true dimensions. The edge line AB will make an angle of 30° with the front vertical plane. Points on the different surfaces of the prism, Fig. 46, are projected to the transparent planes to produce the front and side elevations and the plan view, and when the planes are revolved to one plane the views will occupy the same relative positions shown in Fig. 1, in which the letters on the different views correspond with those on Fig. 46.

The prism is $2\frac{3}{4}$ inches long, 2 inches wide, and 1 inch thick. As the plan view shows true dimensions, this view is drawn first. To draw this view, locate the point B on a horizontal line $2\frac{5}{8}$ inches from the upper border line and $2\frac{1}{2}$ inches from the left-hand border line, then construct a rectangle $ABCD$, $2'' \times 1''$, so that the parallel edges AB and DC make an angle of 30° with the horizontal line as shown. This can be done readily by using a 30° and 60° triangle with the **T** square. From A and B draw AD and BC 1 inch long at right angles to AB and join D and C by a line parallel to AB. Then, $ABCD$ is a rectangle $2'' \times 1''$ and represents the plan, or top view, of the prism.

To construct the front elevation, first draw a horizontal line $5\frac{3}{4}$ inches from the upper border line, as a base line, and $2\frac{3}{4}$ inches above this line draw another horizontal line for the height of the prism. From the corner A of the plan view project a vertical line intersecting the two horizontal guide lines in the points A' and E, and project similar lines from the corners B and C; the lines $A'E$, $B'F$, and $C'G$ thus produced represent the edges of the prism visible in the front elevation. The projection to the point D of the plan view represents an edge that cannot be seen in the front elevation of the prism, therefore it is indicated by a dotted line.

To construct the side elevation, extend the horizontal lines $A'C'$ and EG of the front elevation to the right indefinitely. The widths of the foreshortened surfaces shown in the side elevation are obtained by taking points from the plan view, which is viewed in the direction shown by the arrow. Projectors are drawn with the **T** square from each corner of the plan to intersect a vertical construction line IL, which represents the top edge of the vertical side plane *2–3–5–6*, shown in Fig. 46. The

distances IC and KL are then transferred to the line $A'D''$ with the dividers, thus locating points B'', C'', A'', D'', representing the top edge of the prism in the side elevation. Vertical lines drawn from three of these points to the base line EH' define the visible edges $B''F'$, $C''G'$, and $D''H'$. The dotted line $A''E'$ represents a hidden edge. This will be more easily seen by viewing the plan in the direction indicated by the arrow.

40. Rectangular Prism Parallel to the Front Vertical Plane but Inclined to the Horizontal and Side Vertical Planes.—The drawing of Fig. **2** is of the same prism as that shown in Fig. **1**, but the prism is assumed to have its broad faces parallel to the front vertical plane and the prism is tipped in such a manner that its base makes an angle of 20° with the horizontal plane. The perspective view of the prism shown in Fig. 47 gives a clearer idea of its position. In this illustration the face $abfe$ is parallel to the front edge ij of the horizontal plane $ijkl$, and the edge fg of the prism rests on the surface of the horizontal plane.

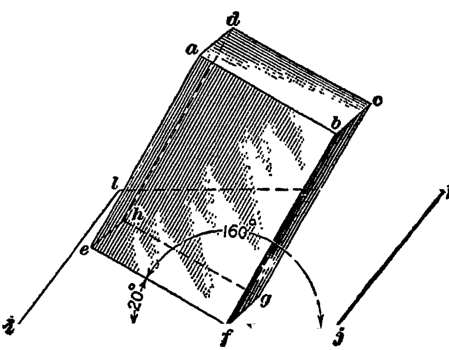

FIG. 47

The front elevation is to be drawn first. This is a rectangle $2\frac{3}{4}$ inches long and 2 inches wide standing with its lower side BA inclined at an angle of 20° to the horizontal plane. To draw this view, locate the point A at a distance of $8\frac{1}{4}$ inches from the left border line. From the point A draw a line at an angle of 20°, using a protractor as shown in Fig. 48. On this line locate the point B at a distance of 2 inches, or the width of the prism. At right angles to the line BA draw lines of indefinite length, and on these lines measure off $2\frac{3}{4}$ inches, or the height of the prism, locating the points C and D, through which draw a line parallel to BA.

To construct the plan, draw vertical projection lines from the points B, C, A, and D upwards as shown. Across them draw a horizontal line $2\frac{1}{8}$ inches from the upper border line, cutting

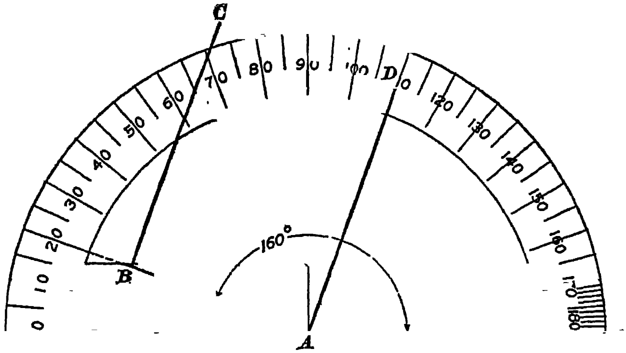

FIG. 48

them in B', C', A', and D', and 1 inch above this draw a second horizontal line cutting them in E, F, G, and H. The edge whose projection is $G\,A'$ is the lowest edge of the prism and cannot be seen from above; hence, $G\,A'$ is dotted in the plan.

To construct the side elevation, transfer the measurement, as $D'\,H$, from the plan to locate the points A'' and G', which are 1 inch apart. From $A''\,G'$ erect two perpendicular lines of indefinite length. Then draw horizontal projection lines from the points $C, D, B,$ and A of the front elevation to intersect these vertical lines. The edge $B''E'$ is hidden in the side view and is therefore indicated by a dotted line.

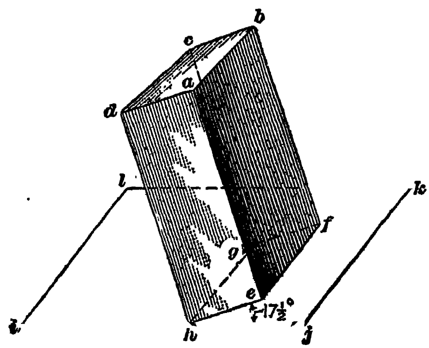

FIG. 49

41. Rectangular Prism Parallel to the Front Vertical Plane but Inclined to the Horizontal and Side Vertical Planes.—The drawing of Fig. 3 is of the same prism as that from which the drawings of Figs. 1 and 2 were

made, but in this case the narrow side of the prism is parallel to the front vertical plane and its base makes an angle of $17\frac{1}{2}°$ with the horizontal. Its position will be more clearly seen from Fig. 49. In this illustration the side *a e h d* is parallel to the front vertical plane, the broad side *a b f e* is inclined to the side vertical plane, and the base *g f e h* is inclined $17\frac{1}{2}°$ to the horizontal.

Draw the front elevation first. This is a rectangle $2\frac{3}{4}$ inches long and 1 inch wide. Locate the corner *A* on the horizontal guide line $4\frac{3}{8}$ inches from the right border line, construct the rectangle at the required angle, and project the plan view and side elevation from the front elevation. No further explanation is necessary, as the views are constructed in a similar manner to those of Figs. 1 and 2.

42. Hexagonal Prism Inclined to the Horizontal Plane but Parallel With the Front Vertical Plane.—In Fig. 4 is shown the projections of a hexagonal prism, which is assumed as being parallel to the front vertical plane and as having the base tilted at an angle of 30° with the horizontal plane.

A plan view of the regular hexagon *A B C D E F* is to be drawn first. Begin the plan by describing a circle with a radius of $\frac{3}{4}$ inch from a center *O* located at a point $4\frac{3}{4}$ inches from the lower border line and $3\frac{1}{2}$ inches from the left border line. Through *O* draw center lines *a b* and *A D* at angles, respectively, of 60° and 30° to the horizontal. Now, with the dividers set to $\frac{3}{4}$ inch, the radius of the circle, lay off the points *A*, *B*, *C*, *D*, *E*, *F*, on the circle for the hexagon, beginning at the point *A*, and by connecting these points complete the plan view.

Begin the elevation by drawing a horizontal line $\frac{3}{4}$ inch above the lower border line and extend it across the sheet. Next, project a line from the corner *D* of the plan parallel to the center line *a b* and intersecting the horizontal guide line in the point *J*. From the point *J* measure off a distance of $2\frac{3}{4}$ inches on this slanting line, locating the point *D'*, which is the height of the prism. Now, from the points *J* and *D'*, draw two lines of indefinite length at an angle of 30° with the

horizontal, and from the plan project the corners A, F, E, and D to the line $G J$ of the elevation, thus producing the edges $A' G$, $F' L$, and $E' K$.

To obtain the side elevation, first draw a series of horizontal projection lines from the points A', F', E', D', G, L, K, and J, and extend them to the right indefinitely. Draw a vertical center line $a' b'$ at a distance of $5\frac{1}{16}$ inches from the left border line. Lay off on each side of the center line $a' b'$ a distance of one-half the width of the prism, measured between parallel sides, thus determining the side edge lines $F'' K'$ and $B' I'$. A plan of the prism may be placed on the center line $a' b'$ from which to project the vertical edge lines. Draw the connecting lines $A'' B'$, $A'' F'$,$'' D'' E'$, etc., from the intersecting points of the vertical and horizontal projection lines, thus completing the view of the top and bottom surfaces of the prism. Corresponding letters are used in the three views, to assist in the reading. The edges $L' G'$ and $G' H'$ are not visible in the side elevation and are therefore shown by dotted lines.

43. Hexagonal Pyramid Whose Axis Is Inclined but Parallel to the Front Vertical Plane.—In Fig. 5 is shown the projections of a regular hexagonal pyramid whose base is tilted at an angle of 30° with the horizontal plane and whose axis $c d$ is parallel to the vertical plane.

The plan, which is a regular hexagon, is constructed about a center O located $4\frac{3}{4}$ inches above the lower border line and 8 inches from the right border line in the same manner that the plan of Fig. 4 was constructed. The size of the hexagon and its position is the same as in Fig. 4. Through the point O draw the center line $c d$ at an angle of 60° and another center line $A D$ at right angles to it. Connect the points A, B, C, D, E, and F, which are the corners of the base, with the center O by lines $A O$, $B O$, etc., thus defining the edges of the intersecting side planes, which are foreshortened when the pyramid is viewed in the direction of the arrow.

To draw the front elevation, locate the point D' on the guide line at its intersection with a line projected from point D of the plan and drawn parallel with the center line $c d$. From this

point D' draw the base line of the pyramid at an angle of 30°, as shown, and project corners A, F, and E from the plan to this base line of the pyramid by lines parallel to the center line $c\,d$. Measure off on the center line $c\,d$ a distance of $2\frac{3}{4}$ inches from the base line $A'\,D'$, thus locating the point O' for the apex of the pyramid. Now, to complete the front elevation, draw lines from the points A', F', E', D' to the point O'.

To construct the side elevation, first locate the vertical center line $c'\,d'$ $6\frac{1}{2}$ inches from the right border line, then from the front elevation draw horizontal lines of indefinite length from the points O', A', F', E', and D' passing through the center line $c'\,d'$. The hexagonal base can be laid off with the dividers from the plan, or the sides may be projected from an auxiliary plan placed in the required position on the center line $c'\,d'$. The lines $F''\,E''$ and $B''\,C''$ representing the parallel sides may now be projected from the plan and the intersecting points E'', D'', C'' and F'', A'', B'' connected by lines, which complete the foreshortened base of the hexagonal pyramid. The elevation is finished by drawing lines from the corners of this base to the apex O''. The dotted lines of the base represent the edges that are not seen.

44. Cylinder Inclined to the Horizontal Plane but Parallel With the Front Vertical Plane.—In Fig. 6 is shown the front and side elevations of a cylinder which is inclined to the horizontal plane at an angle of 30° and is parallel to the vertical plane. The cylinder is $1\frac{1}{2}$ inches in diameter and $2\frac{3}{4}$ inches long. Begin the construction by locating a point i at a distance of $3\frac{1}{2}$ inches above the lower border line and $3\frac{1}{2}$ inches from the right border line. With the bow-pencil set to a radius of $\frac{3}{4}$ inch, describe a circle from the center i for the plan view. Through the point i draw a center line $a\,b$ at an angle of 60° and through the same point draw another center line $c\,d$ at right angles to it, or 30° with the horizontal. From the plan view project two lines from the ends of the diameter on $c\,d$, parallel to the center line $a\,b$, making the line $e\,f$ $2\frac{3}{4}$ inches long, equal to the height of the cylinder, and draw a

line for its base from the point *e* at an angle of 30°, completing the elevation. The center of the plan of the cylinder has been located to coincide with the top edge of the cylinder in the front elevation for convenience in projecting points to the side elevation.

To draw the side elevation, divide half the circumference of the circle, from *g* to *f*, into eight equal parts at *2*, *3*, *4*, etc. From these division points on the circle draw projectors parallel to the center line *a b* to intersect the line *c d*, or the top edge of the cylinder, in points *2'*, *3'*, *4'*, etc. Now draw the vertical center line *h k*, for the side elevation, $1\frac{1}{2}$ inches from the right border line. On this center line describe another circle, from some point *j*, to serve as an auxiliary plan from which to project the diameter of the cylinder. Divide the auxiliary plan into sixteen equal parts in points *1''*, *2''*, *3''*, *4''*, etc. Now, from the division points *1'*, *2'*, *3'*, etc. of the front elevation draw horizontal projectors to the right to meet vertical projectors *1''*, *2''*, *3''*, etc. drawn from the auxiliary plan, and where the vertical and horizontal projectors of similar number intersect, locate points through which a curved line is drawn to form an ellipse. In order to avoid confusion, only a few points on the ellipse are numbered. For example, *4'''* is located in the side elevation where a horizontal projector is drawn from the point *4'* of the front elevation, intersecting a vertical projector drawn from point *4''* of the plan view.

Extreme accuracy is necessary in dividing the circles into similar equal parts. The projectors must be drawn exactly from these points of division, otherwise an imperfect ellipse will be formed. The pencil point should be round and sharp for drawing through the points and around an irregular curve. The foreshortened elliptical base is projected similarly to the top.

DRAWING PLATE, TITLE: CONIC SECTIONS

45. Explanation of Conic Sections.—Curves of different shapes are formed by the intersection of a plane with a circular cone or a cylinder. The intersecting plane, sometimes called a **cutting plane**, is imagined to be a flat surface of indefinite extent and without thickness. The surfaces resulting from the intersection of a cutting plane with a cone or a cylinder will have different outlines, according to the inclination, or slope, of the cutting plane. The curves produced by a plane cutting a cone or a cylinder are known as **conic sections**, which, as shown later, may be circles, ellipses, parabolas, or hyperbolas.

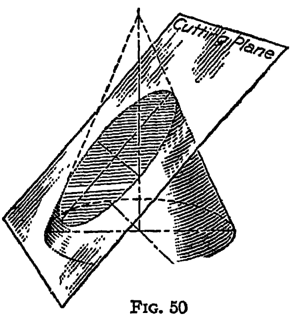

FIG. 50

A **right cone** is one having a circular base and its vertex lying on a vertical line, called the axis, perpendicular to the center of the base. A cone with a circular base and having its vertex not perpendicular to the center of the base is called an **oblique circular cone.**

If a right cone is cut by a plane parallel to its base, that is, by a plane perpendicular to the axis, the section is a circle. If the cutting plane is not perpendicular to the axis and cuts off the entire top of the cone, then the section made is an ellipse. Fig. 50 shows how an ellipse is formed in this manner.

46. Conic Section Forming an Ellipse.—In Fig. 1 are shown the projections of a conic section forming an ellipse. In this case the cutting plane *a b* is inclined at an angle of 52° with the base of a right cone $3\frac{3}{8}$ inches high and 3 inches in diameter at the base. To construct Fig. 1, draw the center line *m n* $2\frac{1}{4}$ inches from the left border line; and at right angles to it draw the center line *p q* at a distance of $2\frac{1}{4}$ inches from the upper border line. Draw a horizontal line of indefinite length for the base of the cone in the elevation. From the point *O*, with a radius of $1\frac{1}{2}$ inches, describe a circle for the

plan view and from the points A and B on the circumference of this circle draw projectors parallel to the center line $m\,n$ to intersect the base line of the cone in the elevation in points A' and B'. Lay off the vertical height of the cone in the elevation from dimensions given and draw the slanting sides of the cone $A'\,O'$ and $B'\,O'$. The cutting plane is represented in the front elevation by a line $a\,b$ drawn at an angle of 52° to the base, by using a protractor as shown in Fig. 51. Divide the circumference of the circle that represents the plan view into any number of equal parts, in this case 24, and from the points of division A, E, H, etc., draw radial construction lines $A\,O$, $E\,O$, $H\,O$, etc. to the center O. Draw also from these points,

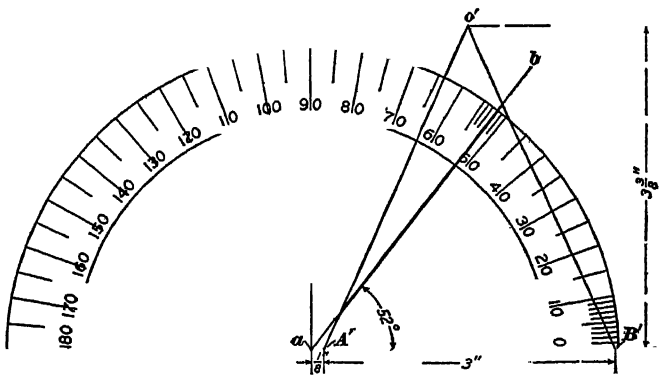

FIG. 51

vertical projection lines, as $A\,A'$, $E\,E'$, $H\,H'$, etc., cutting the base of the front elevation in the points E', H', etc. From these latter points draw lines to the apex O' of the cone, as $E'\,O'$, $H'\,O'$, etc., cutting the line $a\,b$ in the points D', F', etc. From the points D', F', etc. in the elevation draw to the plan the vertical projecting lines $C'\,C$, $D'\,D\,D''$, $F'\,F\,F''$, $L'\,L$, etc., intersecting the radial construction lines $O\,A$, $O\,E$, $O\,H$, $O\,B$, etc. in the points C, D, F, L, F'', etc. Through these points of intersection draw the ellipse by the aid of an irregular curve. It should be observed that points D, D'' in the plan correspond with the point D' in the elevation, and points F, F'' with the point F'.

It will be noticed that in the drawings on this plate the objects are represented as having been cut through by the plane and a portion removed; only the outlines of the part remaining are shown with full lines, all other outlines being dotted.

47. Section Forming a Parabola.—Fig. 52 shows a cone through which a plane has been passed cutting it at an angle parallel to its slanting side *a b*, or, in other words, parallel to one of its elements and intersecting the base. The curve formed by this intersection is a parabola.

In Fig. 2 the plan and front elevation of the cone are located on the center line *m n* $2\frac{1}{4}$ inches from the right border line, and the curve of intersection in the plan is found by the same method as that used in the preceding problem. To find the side elevation of the intersection, proceed as follows:

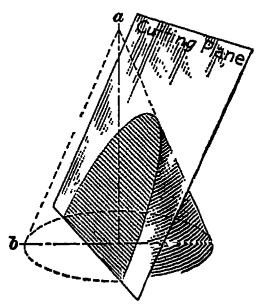

Fig. 52

Draw the side elevation $A'' B'' O''$, which is similar in form to the front elevation, with the center line *t v* as its axis $5\frac{3}{4}$ inches from the right border line. Draw horizontal projection lines from D', F', etc. of the front elevation to intersect the center line *t v* of the side elevation in K' I', etc. With the dividers, transfer the distance $I\,D$, $I\,D''$, etc., from the plan to the side elevation as $I'\,D'''$ and $I'\,D^{iv}$, also transfer $K\,F$ and $K\,F''$ from the plan to the side elevation as $K'\,F'''$ and $K'\,F^{iv}$, thus locating points D''' and D^{iv} and F''' and F^{iv}. In the same way other distances may be transferred and other points located, and a curve may be traced through them. The result will be the side elevation of the cone when cut by a plane parallel to one of its elements. A side elevation of Fig. 1 may be drawn similarly.

48. Conic Section Forming a Hyperbola.—Fig. 53 shows a perspective view of a cone with the cutting plane parallel to its axis. When the cutting plane intersects the base of a cone and is not parallel to any element, the curve of intersection is called a hyperbola.

In Fig. 3 the cone has the same dimensions as in the two preceding problems, and the plan and the front elevation are constructed as before. The projection of the curve in the plan in this particular case, where the cutting plane is parallel to the axis of the cone, is a straight line $a\,b$ in the plan and in the front elevation. The side elevation is found, as in the last problem, by drawing horizontal projection lines from F', D' of the elevation to intersect the center line $t\,v$ of the side elevation. With the dividers, transfer from the plan the distances $I\,D$ and $I\,D''$ to the side elevation as $I'\,D'''$ and $I'D^{iv}$, also transfer IF and IF'' to $K'F'''$ and $K'\,F^{iv}$ of the side elevation, thus locating points D''', D^{iv}, F''', F^{iv}. In the same way, other distances may be transferred and other points located through which a curve may be drawn to obtain the required hyperbola.

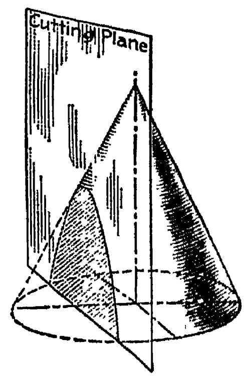

FIG. 53

49. Oblique Section of a Cylinder.—A plane cutting a cylinder at an angle to the axis will produce an ellipse, as shown in Fig. 54. In Fig. 4 is shown the method of determining the ellipse in the side elevation in this case. The cylinder is 2 inches in diameter and $3\frac{5}{8}$ inches long and cut by a plane $a\,b$ making an angle of 57° with the base. Begin by drawing a center line $m\,n$ at $4\frac{1}{4}$ inches from the right border line; and at right angles to it draw the center line $p\,q$ $5\frac{5}{8}$ inches from lower border line. From the point O with a radius of 1 inch, or one-half the diameter of the cylinder, describe a circle for the plan view. Construct the front and the side elevation as already explained, the horizontal projection or plan being a circle having the same diameter as the

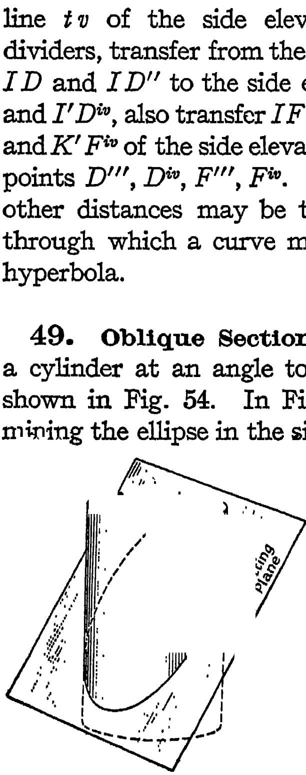

FIG. 54

base. Construct the front elevation from the dimensions given and draw the line $a\,b$ at an angle of 57° to represent the cutting plane.

Draw the side elevation of the cylinder, which is similar in form to the front elevation. To construct the side elevation of the curve, divide the circumference of the circle representing the plan of the cylinder into any number of equal parts, in this case 24, and through the points of division, as A, B, C, etc., draw vertical projection lines as shown. These will cut $a\,b$ in the points A', B', C', D', etc., and horizontal lines are projected from these points to intersect the center line $t\,v$ in the side elevation. This center line is located $1\frac{3}{4}$ inches from the right border line. Then with the dividers transfer from the plan the distances LB, LB''; KC, KC''; ID, ID'', etc. to the side elevation and from the points L', K', I' on the center line $t\,v$ locate the points B''', B^{iv}, C''', C^{iv}, D''', and D^{iv} of the required ellipse. Other points may be transferred and located in like manner, and a smooth curve drawn through the points thus located will be the required side elevation.

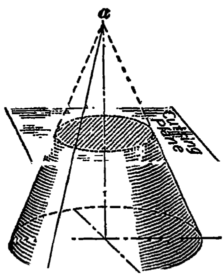

FIG. 55

50. Sections Which Form Circles. When a right cone is cut by a plane parallel to the base of the cone, the resulting section is a circle, as shown in Fig. 55. Its diameter depends on the distance of the cutting plane from the vertex a of the cone. If a cone is cut by a plane through its vertical axis, the resulting section will be an isosceles triangle, as shown in Fig. 56, its outline being the same shape as the front elevation.

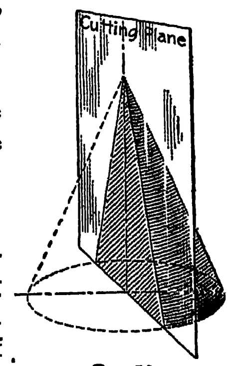

FIG. 56

CONVERTING DECIMALS INTO COMMON FRACTIONS

51. On the next plate some dimensions are given in decimal fractions instead of common fractions. Such decimal dimensions are best laid off with a **decimal scale,** which is a scale with inches divided into tenths, hundredths, etc. Such a scale is not essential for this work, however, and the nearest value of the decimal fraction may be taken in thirty-seconds of an inch.

To change a *decimal fraction* to a *common fraction* having a desired denominator, multiply the decimal by the desired denominator of the common fraction, and express the result as a whole number, which whole number will be the numerator of the fraction.

Thus, to express .765 inch in fourths, we have $.765 \times 4 = 3.06$; placing this product over 4 as a denominator, the result is $\frac{3.06}{4}$ or, say, $\frac{3}{4}$ inch.

To express .765 in thirty-seconds, $.765 \times 32 = 24.48$; with 32 as denominator the result is $\frac{24.48}{32}$ or, say, $\frac{24}{32}$ inch.

If, after multiplying, the decimal part of the product equals .5 or over, drop it and add 1 to the whole number. Thus, in changing a decimal fraction of .66 inch to a common fraction having 16 as a denominator, the result is $.66 \times 16 = \frac{10.56}{16}$, or, say, $\frac{11}{16}$ inch.

The circumference of a circle is equal to the diameter multiplied by 3.1416. Thus, the circumference of a circle with a diameter of $1\frac{3}{8}$ inches is $3.1416 \times 1\frac{3}{8} = 4.32$ inches, or, say, $4\frac{5}{16}$ inches.

The circumference of a circle whose diameter is $1\frac{1}{2}$ inches is $3.1416 \times 1\frac{1}{2} = 4.71 = 4\frac{23}{32}$ inches.

The circumference of a circle whose diameter is $1\frac{1}{4}$ inches is $3.1416 \times 1\frac{1}{4}$ inches $= 3.93$, or, say, $3\frac{15}{16}$ inches.

52. Decimal Equivalents.—Decimal equivalents for the fractions most commonly used by mechanics are given in the

following table. Since the results obtained in calculations are usually decimal quantities, it is desirable that a convenient means should be afforded for their conversion into ordinary fractions. By the aid of this table any decimal may be readily resolved into its corresponding fraction or into a fraction that approaches its real value nearly enough for all practical purposes

8ths	$\frac{1}{8}$	= .125
	$\frac{1}{4}$	= .25
	$\frac{3}{8}$	= .375
	$\frac{1}{2}$	= .5
	$\frac{5}{8}$	= .625
	$\frac{3}{4}$	= .75
	$\frac{7}{8}$	= .875
16ths	$\frac{1}{16}$	= .0625
	$\frac{3}{16}$	= .1875
	$\frac{5}{16}$	= .3125
	$\frac{7}{16}$	= .4375
	$\frac{9}{16}$	= .5625
	$\frac{11}{16}$	= .6875
	$\frac{13}{16}$	= .8125
	$\frac{15}{16}$	= .9375
32ds	$\frac{1}{32}$	= .03125
	$\frac{3}{32}$	= .09375
	$\frac{5}{32}$	= .15625
	$\frac{7}{32}$	= .21875
	$\frac{9}{32}$	= .28125
	$\frac{11}{32}$	= .34375
	$\frac{13}{32}$	= .40625
	$\frac{15}{32}$	= .46875
	$\frac{17}{32}$	= .53125
	$\frac{19}{32}$	= .59375
	$\frac{21}{32}$	= .65625
	$\frac{23}{32}$	= .71875
	$\frac{25}{32}$	= .78125
	$\frac{27}{32}$	= .84375
	$\frac{29}{32}$	= .90625
	$\frac{31}{32}$	= .96875
64ths	$\frac{1}{64}$	= .015625
	$\frac{3}{64}$	= .046875
	$\frac{5}{64}$	= .078125
	$\frac{7}{64}$	= .109375
	$\frac{9}{64}$	= .140625
	$\frac{11}{64}$	= .171875
	$\frac{13}{64}$	= .203125
	$\frac{15}{64}$	= .234375
	$\frac{17}{64}$	= .265625
	$\frac{19}{64}$	= .296875
	$\frac{21}{64}$	= .328125
	$\frac{23}{64}$	= .359375
	$\frac{25}{64}$	= .390625
	$\frac{27}{64}$	= .421875
	$\frac{29}{64}$	= .453125
	$\frac{31}{64}$	= .484375
	$\frac{33}{64}$	= .515625
	$\frac{35}{64}$	= .546875
	$\frac{37}{64}$	= .578125
	$\frac{39}{64}$	= .609375
	$\frac{41}{64}$	= .640625
	$\frac{43}{64}$	= .671875
	$\frac{45}{64}$	= .703125
	$\frac{47}{64}$	= .734375
	$\frac{49}{64}$	= .765625
	$\frac{51}{64}$	= .796875
	$\frac{53}{64}$	= .828125
	$\frac{55}{64}$	= .859375
	$\frac{57}{64}$	= .890625
	$\frac{59}{64}$	= .921875
	$\frac{61}{64}$	= .953125
	$\frac{63}{64}$	= .984375

DRAWING PLATE, TITLE: INTERSECTIONS AND DEVELOPMENTS

53. The drawing plate Intersections and Developments now to be taken up deals with the intersections of the surfaces of solids and the method of obtaining the miter line, as it is commonly called. By the methods explained in connection with this plate the patterns of the intersected surfaces may be developed. One of the important requirements in the construction of these figures is to locate with accuracy the points lying in the line of intersection.

54. Intersection of Two Unequal Cylinders at Right Angles.—Fig. 57 shows a perspective view of two cylinders of different diameters intersecting at right angles to each other. Fig. 1 shows a plan and a front and a side elevation of the object.

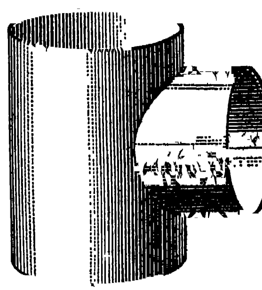

FIG. 57

To draw the views shown in Fig. 1, first locate the axis, or center line *m n*, of the larger cylinder $1\frac{1}{2}$ inches from the left border line. Next draw the horizontal center line *p q* of the small cylinder $1\frac{5}{16}$ inches from the upper border line.

On these center lines, with the dimensions given in the figure, draw a front elevation consisting of the outline *1 A P N L M E F*. The curve *A B C D G Q E*, of course, cannot be drawn yet, as the points through which the curve is to be drawn are still to be determined.

Below the front elevation, draw the plan located on a center line *r s* $4\frac{1}{8}$ inches from the upper border line. To the right of the front elevation draw the side elevation as shown, locating it on a vertical center line *t u* $4\frac{1}{8}$ inches from the left border line. At the intersection of *p q* and *t v*, describe a circle representing the smaller cylinder, and complete the side elevation. Divide the circle in the side elevation into 12 equal parts and from points of division *1*, *2*, *3*, *4*, etc. project horizontal

lines to the front elevation. The points of division *1*, *2*, *3*, *4*, etc. of the circle in the side elevation are to be transferred to the plan view, in the following manner:

Through the point of intersection of the center lines $t\,v$ and $r\,s$ draw a line $x\,y$ at an angle of 45°. Then draw horizontal and vertical construction lines $O\,I$ and $O\,K$ so that they will intersect on the line $x\,y$ at a convenient point, as O.

The points of division on the circle in the side elevation may now be projected to the line $O\,I$ and then each point is to be revolved to the vertical line $O\,K$, O being used as a center. From the points on the line $O\,K$, extend projectors to intersect the large cylinder in points A', B', C', and D' in the plan view. From these points in the plan view, draw vertical projectors that will intersect the projectors *1*, *2*, *3*, *4*, etc. drawn from the side elevation, in points A, B, C, D, etc., through which a curve may be drawn to represent the required intersection of the two cylindrical surfaces; this completes the front elevation. There is, of course, a similar curve from E to A on that part of the object hidden from view.

55. If the part *1* $A\,D\,E\,F$ were assumed to be of thin sheet metal, and if it were cut along the line *1* A and laid out on a flat surface, it would have the form shown in Fig. **2**, and this figure would be termed the **development** of the cylindrical part *1* $A\,D\,E\,F$.

To determine the form and size of this development, draw a horizontal line $6\frac{1}{4}$ inches from the upper border line, and on this line lay off a distance *1–1′*, Fig. **2**, equal in length to 4.32 inches, which is the circumference of the circle shown in the side elevation of Fig. **1** and this circle is the projection of the cylindrical part in question.

Divide the distance *1–1′*, Fig. **2**, into as many equal parts as were used in dividing the circle, in this case twelve, and at these points of division erect perpendiculars to the horizontal line *1–1′*. Then with the dividers measure off the distances *1* A, *7* E, and *1′* A' each equal to *1* A in the front elevation; *2* B, *6* Q, *8* Q' and *12* B' equal to *2* B in the elevation; *3* C, *5* G, *9* G', *11* C' equal to *3* C; and *4* D and *10* D' equal to *4* D

of the elevation. A smooth curve drawn through the points A, B, C, etc. thus found will complete the development.

The form of this development is such that if it were cut out along its bounding line and bent into cylindrical form and joined along the edges $1\ A$ and $1'\ A'$, its irregular edge would fit snugly to the large cylinder as indicated in the front elevation in Fig. 1.

56. Intersection of Two Equal Cylinders at Right Angles.—In Fig. 58 is shown an elbow that consists of two cylinders intersecting each other at right angles. By referring to Fig. 3 it is seen that the lines of intersection between cylinders equal in diameter and intersecting each other in the same plane, so that their axes also intersect, are always represented by straight lines in a view that shows the axes in their true length. Thus, in Fig. 3, the projection of the line of intersection is the straight line $A\ G$.

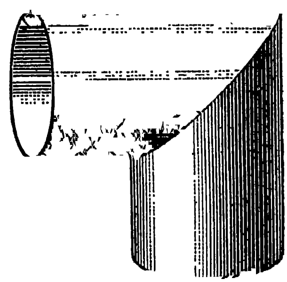

FIG. 58

To obtain the development of the cylindrical part $A\ G\ H\ K$, proceed as follows:

Begin by drawing the front elevation from the dimensions given, locating the horizontal axis $1\frac{9}{16}$ inches from the upper border line and the vertical axis $7\frac{5}{16}$ inches from the left border line. On these axes, draw two cylinders $1\frac{3}{8}$ inches in diameter, and through the intersection of their axes, as at D, draw a line at an angle of 45°. From the point G measure off distances $G\ H$ and $G\ M$ equal to $\frac{7}{8}$ inch, which determines the length of each cylinder. This completes the elevation.

On the vertical center line below the elevation draw a dotted circle to represent the plan. Divide the circle into a number of equal parts, in this case twelve, and from the points of division draw vertical projection lines to meet $A\ G$ at B, C, etc. Extend the line $H\ K$ indefinitely to the right to form the base line of the development, Fig. 4, and on the extended part set off a distance 1–$1'$ equal to 4.32 inches, the circumference of

the dotted circle. Divide this distance into twelve equal parts, also, and erect perpendiculars to the line *1–1′* from the points of division. From *A*, *B*, *C*, etc., Fig. **3**, draw horizontal projection lines intersecting the perpendiculars in Fig. **4** in *A′*, *B′*, *B″*, *C′*, *C″*, etc., and through the points *A′*, *B′*, *C′*, etc. thus found draw a smooth curve. The resulting figure is the required development. Since the part *A G M L* is exactly like *A G H K*, its development is the same as in Fig. **4**. Hence, if two flat thin plates were cut to the same shape and size as this development and each were rolled into true cylindrical form with the edges *1 G′* and *1′ G″* coinciding, they could be put together as shown in the front elevation and the slanting edges would fit accurately at all points.

57. Intersection of Two Unequal Cylinders at an Angle of 65°.—Fig. 59 is an illustration of two cylinders of unequal diameters joining each other at an angle of 65° instead of 90° as in Fig. **1**. It is desired to determine the curve of intersection of the two cylinders in the front elevation. The method of doing this is shown in Fig. **5**.

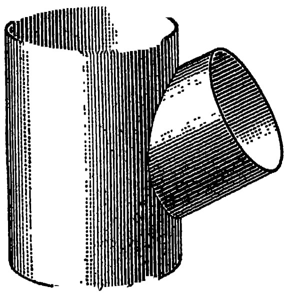

FIG. 59

Begin by locating a vertical center line *m n* $1\frac{1}{2}$ inches from the left border line, and draw a front elevation of the cylinder *L M P N* $1\frac{9}{16}$ inches diameter and $2\frac{1}{8}$ inches long, locating the base line $3\frac{7}{8}$ inches from the lower border line. Below the elevation draw a horizontal center line *r s*, $2\frac{3}{4}$ inches from the lower border line, to intersect the center line *m n*, at which intersection describe a circle $1\frac{9}{16}$ inches in diameter. Draw two horizontal lines of indefinite length $\frac{5}{8}$ inch each side of the center line *r s*, intersecting the circle in *D′* and *D″* in the plan. Project a vertical line from *D′ D″* to intersect the elevation in *S′ S*, and on this line locate point *D* $1\frac{3}{16}$ inches from the line *L M*. By use of a protractor, a center line *p q* for the inclined cylinder may now be drawn from point *D* at an angle of 65° with the axis *m n*. On the

line $p\,q$, measure off $\frac{11}{16}$ inch from the point z and draw line $F\,1$. Two lines drawn from E and A parallel to $p\,q$ complete the elevation, except the curve of intersection, which is projected from the plan view and the dotted circle in the same manner as described for Fig. **1**. The only difference lies in locating the point O from which the projectors are revolved from the dotted circle to the plan view.

To locate the construction lines $O\,I$ and $O\,K$, proceed as follows: At right angles to $p\,q$ draw a center line $t\,u$ to intersect the horizontal center line $r\,s$, thus locating a point V. With the point V as a center, describe two short arcs cutting the center lines $r\,s$ and $t\,u$, and tangent to these arcs draw construction lines $O\,I$ and $O\,K$ parallel to $p\,q$ and $m\,n$, respectively, to intersect in point O. The points of division from the dotted circle may now be projected to the line $O\,I$, and with O as a center the projection lines can be revolved to intersect the line $O\,K$ and then be extended to the plan view. The ellipse in the plan view is the projection of the inclined end of the small cylinder in the front elevation, and its outline is determined by the method used in connection with Fig. **6** of the plate Projections—II.

58. The development of the pattern for the smaller cylinder of Fig. **5** is shown in Fig. **6**. To draw it, measure off on a horizontal line the distance *7–7′* equal to 3.93 inches, equal to the circumference of the dotted circle in Fig. **5**. Divide *7–7′* into twelve equal parts as was the circle, and erect perpendiculars. It is assumed that the cylinder is cut along the line $E\,F$ in the elevation in Fig. **5** and therefore in the development of this part it will be the smallest vertical distance to be measured off.

Begin at the middle point *1* A by making the vertical distance *1* A in the development equal to *1* A of the elevation. Measure off the vertical distances *2* B, *12* B', *3* C, *11* C', etc., equal, respectively, to the distances *2* B, *3* C, etc. of Fig. **5**. Continue this until the two ends are reached in *7* E and *7′* E'. Trace a smooth curve through the points thus found, and the resulting figure is the required development. The development

of the larger cylinder will have a hole of a certain shape due to the small cylinder intersecting it at an angle.

59. Fig. 7 represents the development of the large cylinder shown in Fig. 5 and is obtained as follows: Extend the lines $L\,M$ and $N\,P$ of the elevation indefinitely to the right to indicate the height of the large cylinder in the development. At a distance of $5\frac{1}{2}$ inches from the left border line draw a vertical line intersecting the two extended lines in points X' and Y'. From Y' measure to the right a distance equal to 2.45 inches, locating Y'', and at this point erect a perpendicular to intersect the horizontal line in X''. The horizontal distance $Y'\,Y''$ of Fig. 7 must be taken from the plan, Fig. 5, and is the distance $Y'A'Y''$, which is seen to be one-half the circumference of the circle. Therefore, the rectangle $X'\,X''\,Y''\,Y'$ is equal to one-half the surface of the cylinder of which the circle is the plan.

Next, at the middle point P' of the line $Y'\,Y''$, erect a perpendicular $P'\,M$. From the circle in the plan, with the dividers, measure off the chord distance $A'\,B'$ and transfer this distance along the line $Y'\,Y''$, beginning at P', and locate the point N'. The distances $B'\,C'$ and $C'\,D'$ of the plan are transferred successively in the same manner, locating points R' and S' on the line $Y'\,Y''$. Erect perpendiculars at these points to intersect the line $X'\,X''$ in points N, R, and S. Next, by the use of the **T** square, from the points A, B, C, D, etc., of the elevation in Fig. 5, project construction lines to intersect the perpendiculars $P'\,M$, $N'\,N$, etc., thus locating points A_1, B_1, C_1, etc. If it is desired, these points may be transferred from the elevation with dividers; for example, the distance $P\,A$ in the elevation will equal $P'\,A_1$ in the development, and $N'\,B$, $R'\,C$, and $S'\,D$, etc. in the elevation will equal $N'\,B_1$, $R'\,C_1$, $S'\,D_1$, etc. in Fig. 7. The points A_1, B_1, C_1, etc. having thus been located, similar points to the right of the center line $P'\,M$ are located by the intersection of the same projected horizontal lines and verticals located from the elevation in a manner similar to that employed for locating $N'\,N$, $R'\,R$, etc. A smooth curve may be drawn through the points thus found

and the resulting outline will be the development of the line of intersection of the large cylinder.

If the pattern were cut of the same size and shape as shown in Fig. 7 and rolled to a semicylindrical form, the edges of the opening would coincide with the edges of a cylinder whose development is shown in Fig. 6.

60. Development of Middle Section of Three-Piece Elbow.—Fig. 60 shows a perspective view of an elbow composed of three cylinders of the same diameter and connected so that the two end sections are at right angles to each other.

The diameter of each of the cylinders is $1\frac{1}{2}$ inches. To construct the elevation shown in Fig. 8, draw a horizontal center line $m\,n$ $4\frac{9}{16}$ inches from the upper border line, and at right angles to it draw a center line $p\,q$ 5 inches from the right border line.

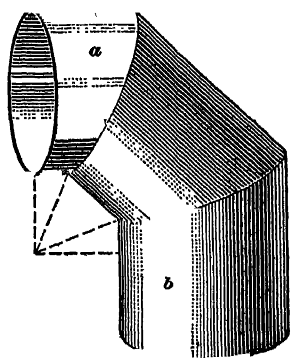

Fig. 60

Where these two center lines intersect in T draw a center line $u\,v$ at an angle of 45°. At a distance of $1\frac{1}{2}$ inches to the left of the center line $p\,q$ draw a vertical line Rx of indefinite length intersecting the line $u\,v$ in the point O.

From the point O, draw a horizontal line OK of indefinite length, and with O as a center and the pencil compasses set to a radius of $1\frac{1}{2}$ inches, describe an arc from I to J.

To determine the joint, or miter, lines AG and MH, bisect the arcs $I\,4$ and $4\,J$ by setting the compasses to any convenient radius and describing short arcs from centers 4 and I, also from 4 and J, and through the intersections of the arcs draw lines OA and OM. Now, on the center line $m\,n$, draw a circle with a radius of $\frac{3}{4}$ inch to represent an end view. From the ends of its diameter project two horizontal lines $R\,A$ and $S\,G$ in the elevation. The point D is located where the center line $m\,n$ intersects the line $O\,A$, and from this point a center line for the middle section may be drawn at an angle of 45°

intersecting the line $O M$ in D', and two lines $A M$ and $G H$ representing the outside of the cylinder are drawn parallel to it. The vertical cylinder section may now be drawn by locating the horizontal line $P N$ $2\frac{1}{4}$ inches below the center line $m n$. The two vertical lines $H P$ and $M N$ are now drawn, completing the elevation.

61. In Fig. **9** is shown the development of the middle section $A G H M$ of Fig. **8**. First divide the dotted circle, Fig. **8**, into twelve equal parts and project the points of division to the elevation to intersect the joint line $A G$, thus fixing the points A, B, C, D, E, etc., and from these points draw lines across the section $A G H M$ parallel to the center line $D D'$, or, in other words, at right angles to the center line $u v$. Where the points of division cross the center line $u v$ locate points *1*, *2*, *3*, *4*, etc.

Then, in Fig. **9**, draw a horizontal line $4\frac{7}{8}$ inches from the lower border line and on this line measure off a distance *1–1'* equal in length to the circumference of the dotted circle in Fig. **8**, which is 4.71 inches. Divide this length into twelve equal parts corresponding to the twelve equal parts into which the dotted circle in Fig. **8** is divided and draw vertical lines through the points of division. Assume that the joint of the middle section is at $G H$ in Fig. **8**. Then in Fig. **9** make $1 G'$ and $1' G''$ equal to $1 G$, Fig. **8**; and $2 F'$ and $12 F''$ equal to $2 F$; $3 E'$ and $11 E''$ equal to $3 E$; $4 D'$ and $10 D''$ equal to $4 D$; $5 C'$ and $9 C''$ equal to $5 C$; $6 B'$ and $8 B''$ equal to $6 B$; and $7 A'$ equal to $7 A$.

A curve through the points G', F', E', D', etc. gives the outline of the upper half of the development. The other half of the curve below the line *1–1'* is of exactly the same shape, and is obtained by transferring the points with the dividers.

62. Curve of Intersection of Flattened Rod.—In Fig. 61 is illustrated a cylindrical rod that is enlarged at the one end, and the sides have been flattened, thus making two parallel faces intersecting a curved surface, and producing a curved line. It is desired to find the exact form of the curve. Fig. **10** shows how this may be done.

The object is a cylindrical rod $1\frac{1}{4}$ inches in diameter and the extreme end has been enlarged to a diameter of $2\frac{7}{8}$ inches. The two parallel faces, as shown in the plan, are $1\frac{5}{8}$ inches apart. It is desired to find the exact form of the curve *A 3 B* representing the intersection of one of the flat surfaces with the curved surface, as shown in the elevation.

First draw a vertical center line *m n* $2\frac{3}{16}$ inches from the right border line, and at right angles to it draw a horizontal center line *E F*, $1\frac{11}{16}$ inches from the upper border line, intersecting at *O*, from which point describe a circle with a $\frac{5}{8}$-inch radius to represent a top view of the rod, which is section-lined to indicate that it is broken. Another circle is described from the center *O* with a radius of $1\frac{7}{16}$ inches for the large diameter. The two flat faces are represented by two horizontal lines *C D* and *A′ B′* drawn at a distance of $\frac{13}{16}$ inch above and below the center line *E F*, thus completing the plan.

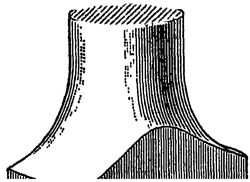

FIG. 61

Begin the elevation by drawing a base line $7\frac{3}{8}$ inches from the upper border line. From the points *E* and *F* of the plan, draw vertical lines to intersect the base line of the elevation in points *E′ F′*. Also project *A′ B′* from the plan to intersect the base line in points *G* and *H*. Measure off a vertical distance of 1 inch from the base line *E′ F′* and draw an indefinite horizontal line. Vertical lines drawn from the base line from points *E′*, *G*, *H*, and *F′* will intersect this horizontal line in points *E″*, *A*, *B*, and *F″*. From the circle in the plan, project two lines for the diameter of the $1\frac{1}{4}$-inch rod in the elevation, and connect it to the enlarged end at the base by arcs *J E″* and *I F″*. The arc *I F″* is drawn tangent to the rod with a radius of $1\frac{1}{8}$ inches; the center from which the arc is described is located as follows: Extend the line representing the diameter of the rod to intersect the horizontal line *E″ F″* in the point *K*. With compasses set to a radius of $1\frac{1}{8}$ inches describe a short arc from the point *K*, cutting the vertical line in *I*. With the compasses kept to the same radius, describe short arcs

with I and F'' as centers and intersecting in K', from which point the arc IF'' can be described tangent to the rod and touching the point F''. The arc JE'' may be drawn in a similar manner.

This completes the elevation, except the curve $A\ 3\ B$, which is yet to be determined.

63. Now, with O as a center and radii of suitable lengths, describe arcs cutting the line $A'\ B'$ in several points, as $1'$, $2'$, $3'$, etc., and continue these arcs until they intersect the horizontal line $E\ F$, thus giving the points 4, 5, 6, etc. Project the points 4, 5, 6, etc. downwards to intersect the arc JE'' in the elevation, thus fixing the points $4'$, $5'$, 6, etc. From the latter points draw horizontal lines, as shown, and from the points $1'$, $2'$, $3'$, etc., on the line $A'\ B'$ in the plan, draw vertical lines downwards to intersect the horizontal lines in points 1, 2, 3, etc. Other points between 3 and A may be found in the same way. Then a smooth curve drawn through A, 1, 2, and 3 will be one-half of the required curve. The other half is exactly like the one just drawn, and may be obtained in a similar manner.

64. Development of Surface of Section of Cone. Fig. **11** shows a cone 3 inches high, 2 inches in diameter at the base, and cut by a plane. Fig. **12** shows the development.

The elevation and the projection of the base of the cone are drawn as shown in Fig. **11.** First locate a horizontal center line $1\frac{5}{8}$ inches from the lower border line, and on it, at a distance of 8 inches from the left-hand border line, draw a vertical line for the base of the cone, which is 2 inches in diameter. From this base line, measure to the left on the center line a distance of 3 inches, the height of the cone, thus locating the point O, from which draw the inclined sides of the cone to intersect the base. The cone is cut off at an angle of 50° with the base line. The projection of the base is now drawn in order to determine the position of the elements on the conical surface.

The plan is represented by the dotted circle and is divided into a convenient number of equal parts, in this case twelve. These points of division, 1, 2, 3, etc., are projected to the base line of the elevation in points $1'$, $2'$, $3'$, $4'$, etc., and projection lines are drawn from points $1'$, $2'$, $3'$, etc. to the point O in the

elevation to represent the elements on the surface of the cone. These elements intersect the cutting plane in points A, B, C, etc. Now begin the development in Fig. **12** by locating point O' $3\frac{3}{8}$ inches from the right border line and $3\frac{1}{16}$ inches from the lower border line. With the pencil compasses set to a radius equal to the slant height of the cone, as $O\ 1'$ or $O\ 7'$, in Fig. **11**, describe an arc 1–$1'$. Make the length of this arc equal to the length of the circumference of the circular base of the cone. This

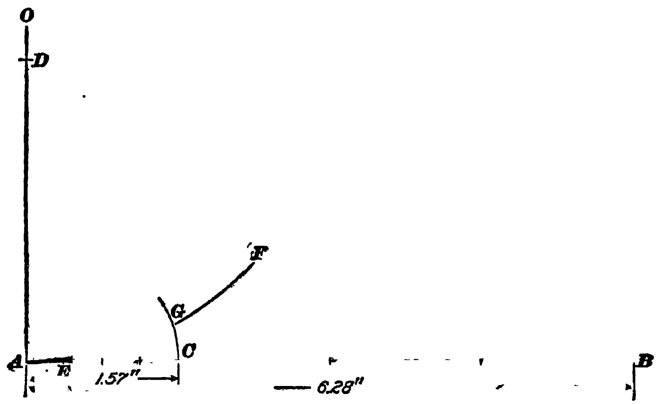

FIG. 62

may be conveniently found as follows: The length of the arc will be $2 \times 3.1416 = 6.28$ inches, nearly. To lay off an arc of this length, the following method may be used:

65. On a separate piece of paper, in the manner shown in Fig. 62, draw a horizontal line $A\ B$, 6.28 inches long, equal to the circumference of the base of the cone shown in Fig. **11**, and erect a perpendicular $A\ O$. Divide the line $A\ B$ into four equal parts, each of the divisions to measure $6.28 \div 4 = 1.57$ inches. Also divide one of these parts, as $A\ C$, into, say, four equal parts.

With the pencil compasses set to a radius of $O\ 1'$, representing the slant height of the cone in the elevation in Fig. **11**, describe a short arc from the point A, cutting the vertical line $A\ O$ in D. With the compasses set to the same radius and with D as a center, describe an arc tangent to the horizontal line $A\ B$. Then, with E as a center and $E\ C$ as a radius,

describe an arc cutting the arc *A F* in *G*; then the arc *A G* is equal in length to the line *A C*, and *A C* is one-fourth the length of the required arc. (Note that, for convenience in this case, Fig. 62 has been reduced to a different scale than that of Fig. 12, which is the reason that measurements in the two do not correspond.)

66. Next, with the dividers set to the chord of the arc *A G* found in the manner just described, begin at point *1* and space off on the arc in Fig. 12 this chord distance four times, marking the points of division between the spaces, as *4*, *7*, *10*, and determining the point *1'*; then the arc *1–1'* will measure 6.28 inches. Each of the four divisions should now be subdivided into three equal parts so that the entire arc is divided into twelve equal parts, as *1*, *2*, *3*, *4*, etc. Join the points of division *1*, *2*, *3*, etc. with the center *O'* by the lines *O' 1*, *O' 2*, *O' 3*, etc., as shown. On the radial lines *O' 1*, *O' 2*, *O' 3*, etc., of Fig. 12, the points that will determine the exact shape of the curve will now be located.

In Fig. 11, *A*, *B*, *C*, *D*, etc. are points of intersection between the radial elements *O 1'*, *O 2'*, *O 3'*, etc. of the cone and the cutting plane, as shown. Only the two outside elements *O 1'* and *O 7'* can be measured directly from this view. The other elements are inclined from the observer, and so do not appear in their true lengths. The line *O D*, for example, if measured on the surface of the actual cone, would evidently be of the same length as the line *O D'*; but in the elevation it is much shorter. Therefore, all points, as *B*, *C*, *D*, etc., must be projected parallel to the base line of the elevation of the cone to intersect the inclined outside element *O 1'* in points B_1, C_1, D_1, etc.

Now continue drawing Fig. 12 by setting off distances *O'A*, *O'B*, *O'C*, etc., equal to the distances *OA*, OB_1, OC_1, etc., taken from Fig. 11, and through these points draw a curve, which will be the desired development of the intersection.

A plate cut of the same size and shape as shown by *A–G–A'–1'–7–1* can be bent into the conical surface shown in the elevation *A–G–7'–1'*.

Particular attention must be given to the method explained above for transferring certain points from the elevation to the

development in Fig. 12. As previously stated, it would be wrong to take measurements from lines *O F*, *O E*, *O D*, etc., as these lines are not represented in their true length until projected to the inclined line *O 1'*, from which the distances are transferred to the development in Fig. 12.

The pattern in Fig. 12 may be laid off, also, by making the radial divisions on the arc *1–1'* equal to the circumference of the cone in Fig. 11 by dividing the distance, which is laid off on a straight line, into as many equal parts as desired. The greater the number of parts into which the line is divided, the nearer these distances approximate a straight line when they are transferred to the arc in Fig. 12.

SHADE LINES

67. The purpose of shade lines on drawings is to show at a glance, without reference to any other view, whether parts represented project above the plane of the surface of the object or whether they are depressions or holes. The value of shade

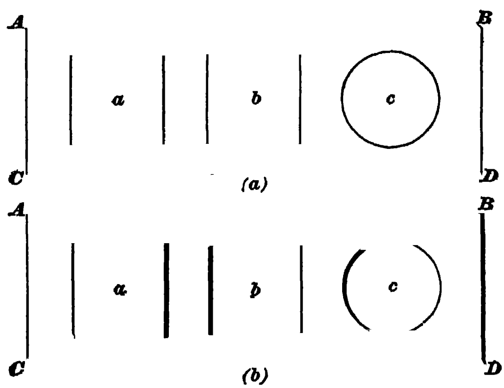

FIG. 63

lines may be seen by comparing the two illustrations shown in Fig. 63 (*a*) and (*b*). In (*a*) the object is rendered with an outline of uniform thickness, and it is not evident whether the shapes of the parts *a*, *b*, and *c* are raised parts or holes. In (*b*)

is shown a drawing of the same object properly shade-lined, which indicates that a is a projecting boss and that b and c represent holes. To determine the height of the bosses, the thickness of the plate $A B D C$, and the depth of the holes, another view would be necessary.

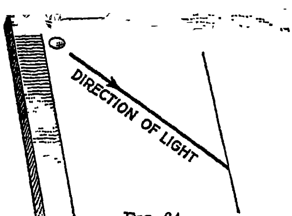

FIG. 64

68. Principles Governing the Location of Shade Lines.—In order that the system of shading may be uniform on all drawings and thus have the same meaning to all, the light is assumed to come in one invariable direction. The light rays are considered as being parallel to the plane of the paper, to make an angle of 45° with all horizontal and vertical lines of the drawing, and to come from the upper left-hand corner of the drawing, as shown in Fig. 64.

A view may be placed in any position on the drawing, but the source of light is always stationary; hence, the lines to be shaded depend on the position of the view. Each view is shade-lined independent of any other. The same principle applies whether the view is a plan or an elevation.

If the rays of light are represented by parallel lines drawn at an angle of 45° with the vertical and horizontal lines, any edge of an object that is touched by such lines is called a light surface; an edge that cannot be touched by lines having this angle is called a dark surface and is represented by a shade line, which is made heavier than the other lines.

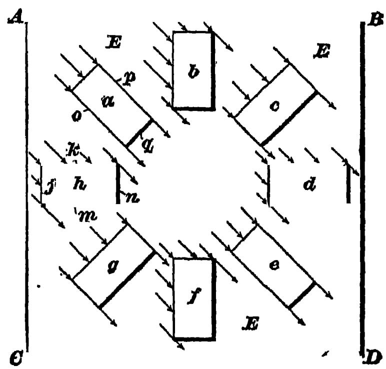

FIG. 65

Fig. 65 shows a square plate $A B D C$, from the surface of which eight rectangular blocks project. The blocks radiate from a center. This example is given to show which edges of the

rectangles should be shade-lined when the rectangles are in different positions. The light is at an angle of 45° and parallel to the plane of the paper, and therefore all surfaces of the object that are parallel to the plane of the paper will be light, such as the surfaces *E* and *a*, *b*, *c*, etc. The light rays, indicated by the arrows, strike certain edges of the projecting rectangles which represent surfaces that are perpendicular to the planes *a*, *b*, *c*, *d*, *e*, *f*, etc.; the edges touched by these light rays therefore remain light edge lines, and the remaining edges of the rectangles are hidden from the light and are shade-lined.

To illustrate more clearly the idea, at *h* the light rays touch the edges *j* and *k*; and edges *m* and *n* are shade-lined, because these edges are not illuminated. When an object is in the position as at *a*, the two sides *o* and *p* are parallel to the light rays and remain as light lines, and the only edge shade-lined is *q*. Sometimes for the sake of appearance the edge *o* is shade-lined; for example, an arm of a pulley may come parallel to the light rays and in this case the draftsman would use his judgment.

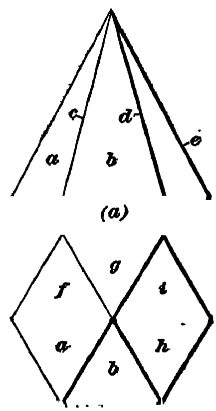

Fig. 66

69. When objects have surfaces that are inclined to the plane of the paper, the edge lines formed by the intersection of these surfaces are shade-lined according to certain rules. For example, in Fig. 66 is shown an elevation and a plan of a hexagonal pyramid.

In the elevation (*a*) the light rays touch the surfaces *a* and *b*, consequently the edge line *c* formed by the intersection of these two planes will be a light line, by the application of the rule that, *Any edge that is formed by the intersection of two light surfaces will be a fine line.* At *d* the edge is shown shade-lined according to an application of another rule that, *All of the edges formed by the intersection of a light and a dark surface are shade-lined.* The edge line *e* is also a shade line, as it is an intersection of two dark surfaces.

In Fig. 66 (*b*) is shown a plan view of the hexagonal pyramid, and this view is shaded without regard to the elevation, it being considered as an independent object and shade-lined accordingly. The surfaces *a*, *f*, *g* are illuminated by the light rays, and the intersection of the light surfaces *a*, *f*, and *g* are light edge lines. The edges between *g* and *i* and *a* and *b* are the intersections of light and dark surfaces and require shade lines. The surfaces *i*, *h*, and *b* are dark surfaces and the intersections between them are shade-lined.

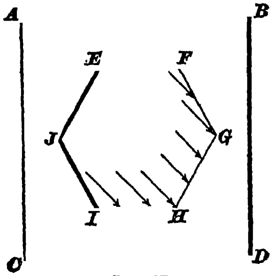

Fig. 67

70. In shading holes or depressions, a slightly different assumption is made in the direction of the light. If the light, in the example shown in Fig. 67, passed over the surface of $ABDC$ parallel to the plane of the paper, as previously assumed, all of the inside surfaces would be dark and the entire outline $EFGHIJ$ would be shaded. In order to prevent this and to make the work appear similar to that which has preceded, the rays of light are assumed to make an angle of 45° with the plane of the paper. In Fig. 68 is shown an illustration of how the light rays are assumed to strike the edges of all holes and depressions. Hence, in Fig. 67 the light will strike the surfaces whose edges are FG, GH, and HI as shown by the arrows, leaving the surfaces whose edges are EF, EJ, and JI dark as before. Therefore, these latter edges will be shaded, and the edges FG, GH, and HI will be light.

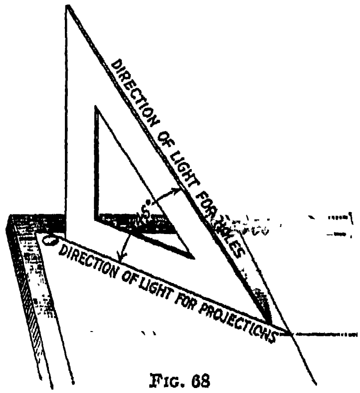

Fig. 68

71. In Fig. 69 is shown the method by which shade lines are added to two concentric circles to convey the idea of a raised surface and also of a circular hole.

After inking the outer or large circle with the pen compasses, draw a pencil line *a b* through its center at an angle of 45°, which is the angle at which the light rays are assumed to illuminate the edges. At right angles to this line and through the center *x* of the circle draw another pencil line. With the compasses set to the same radius, shift the instrument to the point *y* and draw a semicircle that will blend into the outline at the points *c* and *d*. The light rays illuminate the edge *c i d*, which is one-half of the circle,

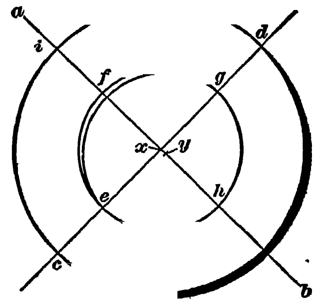

FIG. 69

and the remainder is in shade. The inner circle is now inked in, and the compasses, kept to the same radius, are shifted to center at the point *y*, and from this point another semicircle is drawn which will form a shade line at *e f g*; the edge *e h g* will be a fine line, as that part of the circle is illuminated by the light rays.

There are now two crescent-shaped spaces which are to be filled in with a lettering pen to bring out the effect of the shade line. The shade line of the larger circle is shown filled in.

The thickness of the shade lines is left to the draftsman's judgment. The compasses can be shifted along the line *a b* to any desired distance, but the thickness of these shade lines should correspond with those on other parts of the drawing. When shifting the compasses to a new center, care must be taken to do this accurately on the line, otherwise the semicircles will not be tangent at the desired points.

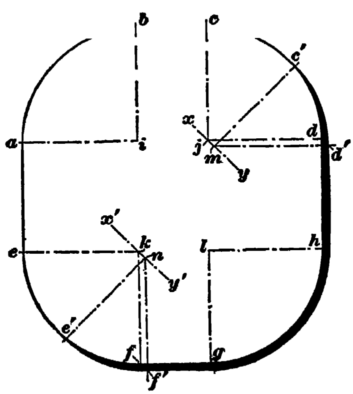

FIG. 70

72. In Fig. 70 is shown the method of shade-lining an object having rounded corners connected by straight lines at right angles to each other.

The arcs $a\,b$, $c\,d$, $e\,f$, and $g\,h$ are inked first, care being taken that the arcs do not extend beyond the points of tangency, and the straight lines are then drawn to meet them. The centers from which these arcs are described are indicated at i, j, k, and l. The main purpose of the illustration is to show how the center point is shifted to describe the required shade lines to conform to the principles of light and shade as already given. The arc $a\,b$ and the straight lines $a\,e$ and $b\,c$ are light edges, but arcs $c\,d$ and $e\,f$ are partly shade-lined.

To add the shade lines on the arcs $c\,d$ and $e\,f$, the compasses are set to the radius of the arc, and the centers are shifted from j and k along the 45° lines $x\,y$ and $x'\,y'$ to the points m and n, respectively, and arcs are drawn tangent to arcs $c\,d$ and $e\,f$, from the points c' and e' to d' and f'.

The arc $g\,h$ is shade-lined without changing the center, by adjusting the pen to the desired thickness of shade line. The shade lines on the straight edges are then added.

Experienced draftsmen save time by shade-lining at the same time the work is being outlined.

Shade lines are extensively used on mechanical drawings in technical publications, as the details can be made clearer and fewer views of the subject are necessary. Shade lines are required on patent-office drawings.

THE TRANSITION SPIRAL

PRELIMINARIES

SUPERELEVATION OF OUTER RAIL

1. As explained in *Kinematics and Kinetics*, a train moving in a circular track exerts a centrifugal force on the rails that is directly proportional to the square of the velocity and inversely proportional to the radius of the curve. As the radius of the curve is inversely proportional to the degree of curve, the centrifugal force is directly proportional to the degree of curve. In order to avoid all danger of derailment, as well as the wear of both rail and wheel caused by their mutual pressure, the outer rail of the curve is **superelevated**; that is, made higher than the inner in such manner that the centripetal force necessary for the circular motion of the train will be furnished by a component of the normal pressure of the rails, and that there will be, therefore, no lateral pressure between rails and wheels. The difference e between the elevation of the outer rail and that of the inner is called the **superelevation** of the outer rail; and, in order that the conditions specified may be fulfilled, the following equation must be approximately satisfied (see *Kinematics and Kinetics*):

$$e = \frac{G' v^2}{g R} \qquad (1)$$

in which e = superelevation, in feet;
G' = horizontal distance, in feet, between centers of heads of rails;
R = radius of curve, in feet;

v = velocity of train, in feet per second;

g = acceleration due to gravity = 32.16 feet per second.

The gauge on curves is a little greater than on level track; for a gauge of 4 feet $8\frac{1}{2}$ inches, G' is approximately 4.9 feet. If the degree of curve is denoted by D, we have, very nearly, $R = \frac{5{,}730}{D}$. If the velocity, in miles per hour, is denoted by V, then

$$v = \frac{5{,}280}{60 \times 60} V = \frac{22}{15} V$$

Substituting these values in formula **1**, and reducing,

$$e = .000058\, D\, V^2 \qquad (2)$$

2. Table I, at the end of this Section, gives the values of e corresponding to all values of V and D that are likely to be required in practice. This table is computed from a more accurate formula than the one just given. The latter formula is sufficiently exact if no tables are at hand.

Table I contains the theoretically best superelevations computed to accommodate the fastest train that will pass over the line. In practice, these figures are often modified, especially if the track is used for both freight and passenger traffic. A superelevation of more than 8 inches is rarely allowed in this case, for this would endanger the slower-moving freight trains. For sharp curves, slow-down orders must be issued. Some railroads make use of special tables of superelevations based on the kind of traffic to be accommodated.

EXAMPLE.—To find the superelevation for a 6° circular curve, if the velocity of the fastest train that is to pass over the curve is 50 miles per hour.

SOLUTION.—In Table I, in the same horizontal line with 6°, and in the column headed 50, we find .844. The superelevation is, therefore, .844 ft. Ans.

EXAMPLES FOR PRACTICE

Find, by Table I, the superelevations for the following circular curves and maximum train speeds:

1. $D = 4°$; $V = 60$. Ans. .811 ft.
2. $D = 10°$; $V = 40$. Ans. .898 ft.
3. $D = 18°$; $V = 30$. Ans. .906 ft.
4. $D = 45°$; $V = 15$. Ans. .559 ft.

DEFINITIONS

3. Transition Curves.—From Arts. **1** and **2**, it is seen that the entire outer rail of a circular curve from the P. C. to the P. T. should be elevated above the inner rail. Theoretically, it should attain this superelevation exactly at the P. C., for as long as the train is on the tangent the two rails should be at the same level, while the instant it enters the curve it should find the outer rail elevated.

It is, of course, impossible to make a sudden change in the elevation of the outer rail exactly at the P. C. It is, therefore, customary to begin to raise the outer rail at a point on the tangent back of the P. C., and gradually to increase the elevation until, at some point on the curve, the outer rail reaches the proper superelevation.

The objections to this method are that the superelevation on the tangent is not theoretically needed, while the proper superelevation is not given to the outer rail throughout the curve. When the train enters a curved track constructed in this way, a disagreeable lurch is felt. Then, too, in passing from a tangent to a curve, the trucks must take a new position with reference to the car body, and the couplers must move through an angle. With long cars and sharp curves, these motions are made very suddenly.

4. In order to overcome the shock and disagreeable lurch of trains due to the sudden change of direction and to the sudden change in elevation of the outer rail, a curve, called a **transition curve,** is introduced, the purpose of which is to connect the tangent with a circular curve in such a manner that the change from one to the other will take place gradually.

5. Center of Curvature.—Let P, Fig. 1, be a point on any curved line APB. From P draw two chords PM and PN, and at the middle points of these chords erect the perpendiculars LO' and HO'. The point O' in which these two lines meet is equally distant from the points M, P, and N; therefore, with O' as a center and a radius $O'P$, a circular arc CPR may be drawn that will pass through M, P, and N.

If, now, the points M and N are moved continually nearer to P, it is found that, as the chords shorten, the intersection O' approaches a definite limiting position O; this point, which O' approaches the more closely the shorter the chords are drawn,

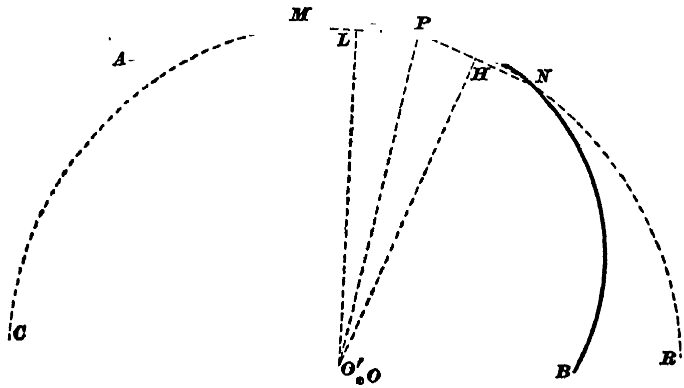

FIG. 1

is called the **center of curvature** of the curve AB at the point P. Its exact position is determined by the use of advanced mathematics; its approximate position may be found by drawing from P two very short chords, and finding the point of intersection of the perpendiculars at their middle points.

6. Osculating Circle and Radius of Curvature.—A circle drawn from O, Fig. 1, as a center, with OP as a radius, is called the **osculating circle** to the curve AB at the point P. This circle is tangent to the curve at P, and its radius OP is called the **radius of curvature** of the curve AB at the point P. The value of the radius of curvature at any

point of a curve can be determined by formulas given in works on the differential calculus.

Fig. 2 shows the osculating circles drawn at three points P_1, P_2, and P_3 of the curve CP_1P_2B. It should be noted that, as the curve becomes sharper, the radius of the osculating circle diminishes.

The **curvature**, or sharpness, of a curve is greater the smaller the radius of curvature. Thus, in Fig. 2, the curve CB

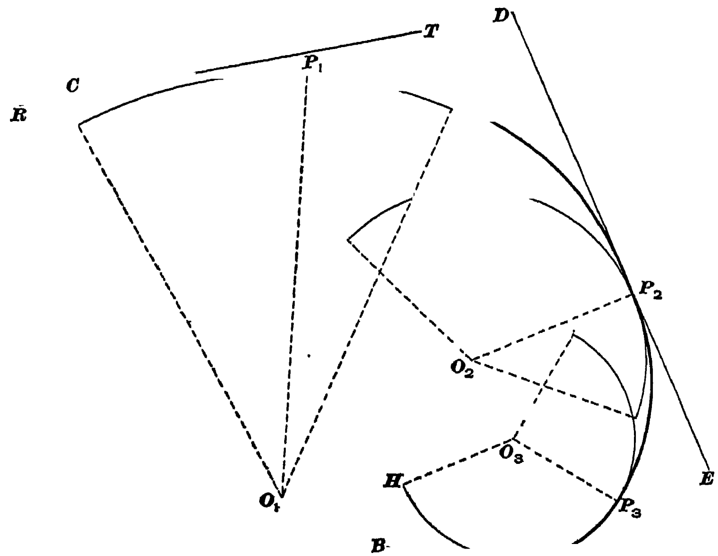

FIG. 2

has a greater curvature at P_3 than at P_2, and a greater curvature at P_2 than at P_1.

7. The **tangent** to any curve at any given point is the tangent to the osculating circle at that point. Thus, in Fig. 2, the tangent to the curve CB at P_2 is the line DE, tangent to the osculating circle O_2.

8. Degree of Curve.—The **degree of curve** *of a simple circular curve* is the angle between two radii drawn from the center to points on the curve 100 feet apart, the distance being measured on one or more chords. Thus,

if LM, Fig. 3, or $LM' + M'M$, Fig. 4, or $LM' + M'M'' + M''M''' + M'''M$, Fig. 5, is 100 feet, the degree of curve is the angle D.

In the best railroad practice, circular curves up to a 7° curve are measured with 100-foot chords, as shown in Fig. 3; from 7° to 14°, they are measured with 50-foot chords, as shown in Fig. 4; and from 14° upwards, they are measured with 25-foot chords, as shown

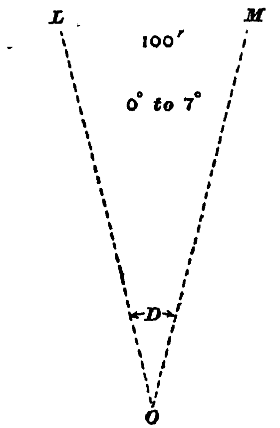

FIG. 3

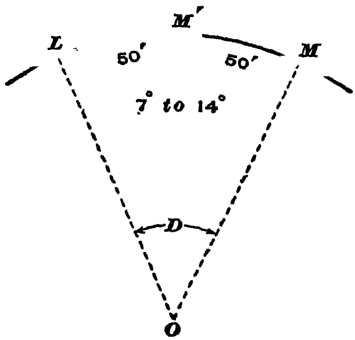

FIG. 4

in Fig. 5. Thus, an 8° curve is one in which two 50-foot chords together subtend an angle of 8° at the center; a 20° curve is a curve in which four 25-foot chords subtend an angle of 20° at the center. When the curve is measured in this way, the formula

$$R = \frac{5{,}730}{D}$$

will give the value of the radius with sufficient accuracy. It is evident that the greater the degree of curve, the greater is the curvature or sharpness of the curve.

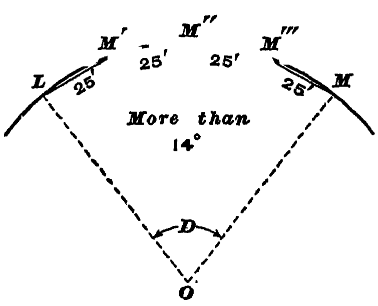

FIG. 5

9. The **degree of curve** *of any curve* at any given point is the degree of curve of a circular curve whose radius is equal to the radius of curvature, at the given point, of the

curve under consideration. Thus, the degree of curve of CB, Fig. 2, at P_1 is the degree of curve of a circular curve whose radius is $P_1 O_1$, the radius of curvature of the curve at P_1.

10. The curvature of a transition curve gradually increases from the P. C., where it is zero, to the point at which the transition and the circular curve meet, where it is equal to the curvature of the latter curve.

Thus, if CP_3, Fig. 2, is a transition curve connecting the tangent RT at C with a circular curve $P_3 H$ at P_3, the degree of curve of CP_3 gradually increases between C, where it is zero, and P_3, where it is equal to the degree of curve of the circular curve $P_3 H$. The circular curve at P_3 is a part of the osculating circle to the transition curve at the same point.

11. The Transition Spiral.—The **transition spiral** is a transition curve in which the degree of curve at any point

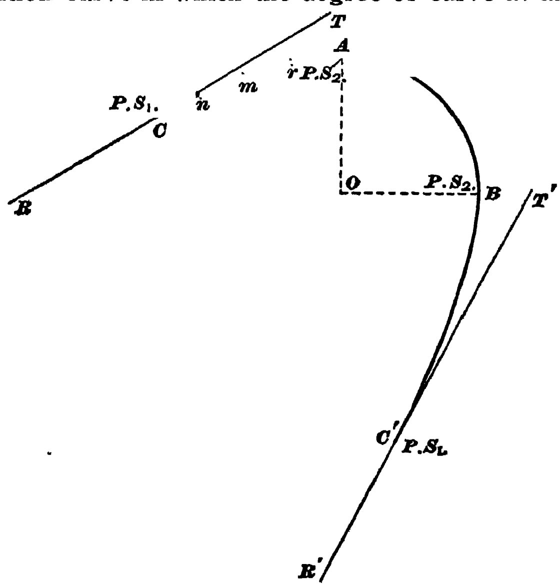

FIG. 6

increases directly as the distance of this point, measured along the curve, from the tangent. The degree of curve is always

zero at the tangent, and, at the point at which the spiral meets the circular curve, equals the degree of the circular curve.

The point at which the transition spiral joins the tangent is called the **point of spiral,** and is denoted by P. S_1 (see Fig. 6).

The point at which the transition spiral joins the circular curve is called the **second point of spiral,** and is denoted by P. S_2.

In Fig. 6, CA is a transition spiral connecting the tangent RT with the circular curve AB: C is the point of spiral P. S_1, and A is the second point of spiral P. S_2. If another spiral is inserted between the end B of the circular curve and the tangent $R' T'$, then, for this curve, B is the second point of spiral, and C' is the point of spiral.

The degree of curve of CA is zero at C, and at A it is equal to the degree of curve of AB. Between these points, it varies as the distance from C. For example, if AB is an 8° curve, the degree of curve of the spiral at a point m, half way between C and A, is $\frac{1}{2} \times 8° = 4°$; at n, one-fourth of the distance from C to A, it is $\frac{1}{4} \times 8° = 2°$; at a point r, three-fourths of the distance from C to A, it is $\frac{3}{4} \times 8° = 6°$.

As with simple circular curves, a distance of 100 feet on the spiral is called a **station.**

GENERAL PROPERTIES OF THE TRANSITION SPIRAL

12. Degree of Curve at Any Point on the Spiral. The **unit degree of curve of spiral** is the degree of curve of the spiral at a point 100 feet from the point of spiral. In Fig. 6, if Cn is 100 feet, the unit degree of curve of spiral is the degree of curve of the spiral at n. In the example given in Art. **11**, the unit degree is 2°.

Let a = unit degree of curve of spiral;
l = distance, in stations, from point of spiral to any other point of the curve;
d = degree of curve of spiral at this point.

From the definition of a transition spiral, it follows that

$$d = a\,l \qquad (1)$$

At the second point of spiral P. S_2, the degree of curve of spiral equals the degree of curve D of the simple circular curve. If L is the whole length of the spiral, expressed in stations, formula **1** becomes

$$D = aL \qquad (2)$$

from which

$$a = \frac{D}{L} \qquad (3)$$

The last formula gives the unit degree of curve of spiral when the degree of the circular curve and the length of spiral are known.

Example 1.—If CA, Fig. 6, is a transition spiral 400 feet long connecting the tangent with an 8° curve, and if n, m, and r are distant 1, 2, and 3 stations, respectively, from P. S_1, find the degree of curve of spiral at n, m, r, and A, the unit degree of curve of spiral being 2°.

Solution.—Applying formula **1**, we have:

$$\left.\begin{array}{lll} \text{At } n, & l = 1 \text{ sta.}; & d = 1 \times 2^\circ = 2^\circ \\ \text{At } m, & l = 2 \text{ sta.}; & d = 2 \times 2^\circ = 4^\circ \\ \text{At } r, & l = 3 \text{ sta.}; & d = 3 \times 2^\circ = 6^\circ \\ \text{At } A, & L = 4 \text{ sta.}; & D = 4 \times 2^\circ = 8^\circ \end{array}\right\} \text{Ans.}$$

EXAMPLE 2.—A spiral 360 feet long connects a tangent with a $4° 30'$ curve. What is the unit degree of curve of spiral?

SOLUTION.—Here, $D = 4° 30' = 4.5°$; $L = 3.6$ sta.; hence, by formula **3**,

$$a = 4.5° \div 3.6 = 1.25° = 1° 15'. \text{ Ans.}$$

EXAMPLES FOR PRACTICE

1. A spiral 240 feet long connects a tangent with a 3° circular curve. Find: (*a*) the unit degree of curve of spiral; (*b*) the degree of curve of spiral at points 50, 150, 200, and 240 feet from the P. S_1.

Ans. $\begin{cases} (a)\ 1° 15' \\ (b)\ 37.5',\ 1° 52.5',\ 2° 30', \text{ and } 3° \end{cases}$

2. In the following two examples, find a for each spiral, and also the values of d at the points indicated: (*a*) Spiral 600 feet long connecting with a 2° curve; points 100, 200, 300, 400, 500, and 600 feet from P. S_1. (*b*) Spiral 150 feet long connecting with a 6° curve; points 20, 30, 60, and 150 feet from P. S_1.

Ans. $\begin{cases} (a)\ a = 20';\ d = 20',\ 40',\ 1°,\ 1° 20',\ 1° 40', \text{ and } 2° \\ (b)\ a = 4°;\ d = 48',\ 1° 12',\ 2° 24', \text{ and } 6° \end{cases}$

13. Superelevation of Outer Rail at Any Point of the Spiral.—The superelevation at any point of the spiral should equal the superelevation required by the osculating circle at that point. The degree of curve of the osculating circle, which is the same thing as the degree of curve of spiral at the point, may be computed by formula **1**, Art. **12**, and the corresponding superelevation computed by the formula in Art. **1**, or taken from Table I at the end of this Section. Another method, which is usually simpler, is as follows: If e is the superelevation at the P. S_2, and D is the degree of the circular curve, we have (Art. **1**),

$$e = .000058\ D V^2$$

Writing $a L$ for D (Art. **12**),

$$e = .000058\ a L V^2 \qquad (1)$$

Similarly, if e_l is the superelevation at any point whose distance from P. S_1 is l stations,

$$e_l = .000058\ a l V^2 \qquad (2)$$

Therefore, dividing (2) by (1), $\frac{e_l}{e} = \frac{l}{L}$

and $$e_l = e \times \frac{l}{L}$$

By this formula, e_l can be computed for any point on the spiral, when e has been computed or taken from the table.

EXAMPLE.—A spiral 400 feet long connects with a 6° curve. To find the superelevation at points 50 feet apart on the spiral, if a train speed of 50 miles per hour is to be allowed for.

SOLUTION BY TABLE.—The unit degree of curve of spiral is first found. Here, $D = 6°$, and $L = 4$ sta.; therefore, by formula **3**, Art. **12**,

$$a = 6° \div 4 = 1° 30'$$

Next, the degree of curve of spiral at each point is found by formula **1**, Art. **12**:

At first point,	$l = \frac{1}{2}$;	$d = 1° 30' \times \frac{1}{2} = 0° 45'$
At second point,	$l = 1$;	$d = 1° 30' \times 1 = 1° 30'$
At third point,	$l = \frac{3}{2}$;	$d = 1° 30' \times \frac{3}{2} = 2° 15'$
At fourth point,	$l = 2$;	$d = 1° 30' \times 2 = 3° 0'$
At fifth point,	$l = \frac{5}{2}$;	$d = 1° 30' \times \frac{5}{2} = 3° 45'$
At sixth point,	$l = 3$;	$d = 1° 30' \times 3 = 4° 30'$
At seventh point,	$l = \frac{7}{2}$;	$d = 1° 30' \times \frac{7}{2} = 5° 15'$
At P. S_2,	$l = 4$;	$d = 1° 30' \times 4 = 6° 0'$

From Table I, the following superelevations are obtained, using interpolation when necessary:

At first point,	$\frac{45}{60} \times .143$	= .107 ft.	Ans.
At second point,	. . . .	= .214 ft.	
At third point,	$.285 + \frac{1}{4} \times (.427 - .285)$	= .321 ft.	
At fourth point,	. . . .	= .427 ft.	
At fifth point,	$.427 + \frac{3}{4} \times (.568 - .427)$	= .533 ft.	
At sixth point,	$.568 + \frac{1}{2} \times (.707 - .568)$	= .638 ft.	
At seventh point,	$.707 + \frac{1}{4} \times (.844 - .707)$	= .741 ft.	
At P. S_2	. . . .	= .844 ft.	

SOLUTION BY FORMULA.—From Table I, the value of e at P. S_2 is found to be .844 ft. The superelevation at the other points is found by the formula $e_l = e \times \frac{l}{L}$. At P. S_1, $e_l = 0$.

At first point,	$e_l = .844 \times \frac{1}{8} = .106$ ft.	Ans.
At second point,	$e_l = .844 \times \frac{1}{4} = .211$ ft.	
At third point,	$e_l = .844 \times \frac{3}{8} = .317$ ft.	
At fourth point,	$e_l = .844 \times \frac{1}{2} = .422$ ft.	
At fifth point,	$e_l = .844 \times \frac{5}{8} = .528$ ft.	
At sixth point,	$e_l = .844 \times \frac{3}{4} = .633$ ft.	
At seventh point,	$e_l = .844 \times \frac{7}{8} = .739$ ft.	

The results differ slightly from those of the first solution, because the formula given in Art. 1 is only approximate. The differences, however, are unappreciable in practical work.

EXAMPLES FOR PRACTICE

Find by Table I the superelevation at each station of the following three spirals:

1. Length = 400 feet; degree of circular curve = 2°; greatest train velocity = 60 miles per hour.

Ans. $\begin{cases}\text{At P. S}_1\text{, 0; at Sta. 1, .103; at Sta. 2, .206;} \\ \text{at Sta. 3, .308; at P. S}_2\text{, .41 ft.}\end{cases}$

2. Length = 500 feet; D = 10°; greatest train velocity = 40 miles per hour. Ans. .000, .183, .365, .545, .723, and .898 ft.

3. Length = 240 feet; D = 6°; greatest train velocity = 60 miles per hour. Ans. .000, .511, 1.006, and 1.196 ft.

14. Advantages of the Transition Spiral.—The student is now prepared to understand the advantages gained by the use of the transition spiral. On leaving the tangent RC, Fig. 6, the train passes over a curve whose curvature continually becomes sharper and sharper until finally it becomes equal to that of the circular curve AB. At each point of the track CA, the superelevation is exactly what it should be to correspond to the curvature, and besides this the direction of motion is changed gradually and uniformly from the direction on the tangent to that on the curve AB. This insures smooth riding and greatly lessens lurching and jarring, which are so injurious to the roadbed and rolling stock. While it is true that the superelevation must be computed for the maximum train speed, yet transition spirals are advantageous for low speeds also.

15. Although a great number of American roads employ transition curves, there are still many lines on which they have not come into use. It may, therefore, not be out of place to state one or two instances in which their value has been clearly shown.

There is a curve on the Lehigh Valley Railroad over which the Black Diamond Express now runs sometimes as

fast as 75 miles per hour. The curve is a 4° curve, 1,000 feet long, at which for many years all trains were obliged to slow down. It was seriously contemplated to change the line to lighter curvature at a cost of about $1,500. Instead of this, a spiral 240 feet long was inserted, when for the first time in 25 years a slow-down was not required for this curve.

An exactly similar result was attained on the Cleveland, Cincinnati, Chicago & St. Louis Railroad by spiraling the numerous curves near Sydney, Ohio. These were on a down grade and the trains were obliged to reduce speed greatly before the spirals were inserted; afterwards, the speed was not reduced.

The cost of spiraling curves on new track is scarcely appreciable, the only expense being the slight additional time required from the engineer. When properly chosen spirals are inserted in old track, the rails need to be thrown but little, if any, more than would be necessary in realinement without spirals. It has even been the experience in some divisions that the cost of maintenance of track was actually less during the years in which the curves were being spiraled than in the years preceding.

16. Angle of Deviation and Angle of Deflection. Let CA, Fig. 7, be a spiral connecting the tangent RT with the circular curve AB. Let P be any point on the spiral and HN a tangent to the spiral at the point P.

The angle that a tangent drawn to the spiral at any point P forms with the original tangent RT is called the **deviation angle** for the point P. It is represented by the Greek small letter δ (called *delta*).

In the figure, the angle HNT is the deviation angle for the point P.

If l is the distance CP expressed in stations, and a is the unit degree of curve of spiral, then

$$\delta = \tfrac{1}{2} a l^2 \qquad (1)$$

This and other formulas relating to the spiral are here given without deriving them, as their derivation requires the use of advanced mathematics.

When the point P coincides with the P. S.$_2$, the deviation angle becomes LKT, which is represented by the Greek capital letter Δ (called *delta*). If L is the whole length CA of the spiral, formula **1** becomes, for the P. S.$_2$,

$$\Delta = \tfrac{1}{2} a L^2 \qquad (2)$$

Since $LKT = AEC$, it follows that Δ is the whole **central angle** of the spiral, which measures the whole change

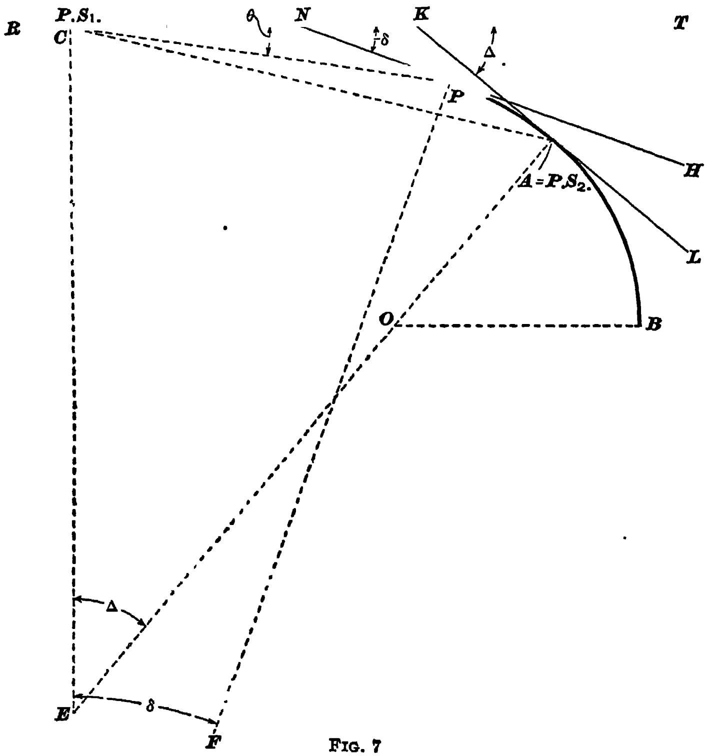

FIG. 7

in direction of the track between the original tangent and the P. S.$_2$.

17. The angle between the original tangent and a chord drawn from the P. S.$_1$ to any point of the spiral is called the **deflection angle** to this point. It is represented by the Greek letter θ (called *theta*). In Fig. 7, TCP is the

deflection angle for the point P. It is the angle that must be deflected at the P. S_1 from the original tangent, in order to locate the point P of the spiral.

It can be shown that θ is slightly less than $\frac{1}{3}\delta$. The formula connecting θ and δ is

$$\theta = \tfrac{1}{3}\delta - N$$

in which the value of N may be taken from the accompanying small table. Intermediate values may be found by interpolation.

$\frac{1}{3}\delta$ Degrees	N Minutes	$\frac{1}{3}\delta$ Degrees	N Minutes
3	.0	8	.7
4	.1	9	1.0
5	.2	10	1.4
6	.3	11	1.9
7	.5	12	2.4

EXAMPLE.—A spiral 600 feet long connects a tangent with a 12° curve. To find the deviation and deflection angles for a point 580 feet from the P. S_1.

SOLUTION.—By formula **3**, Art. **12**, the unit degree of curve of spiral is $12° \div 6 = 2°$. Since the point is 5.8 sta. from the P. S_1, $l = 5.8$. Therefore, by formula **1**, Art. **16**,

$$\delta = \tfrac{1}{2} \times 2° \times 5.8^2 = 33.64° = 33° \; 38.4'. \quad \text{Ans.}$$

To find the deflection angle, we have,

$$\tfrac{1}{3}\delta = 11° \; 12.8'$$

Interpolating from the table just given,

$$N = 1.9' + \tfrac{12}{60} \times (2.4' - 1.9') = 2.0'$$

Therefore, $\theta = \frac{1}{3}\delta - N = 11° \; 10.8'$ Ans.

18. Tables of Transition Spirals.—For laying out a spiral in the field, certain angles and distances are required. These may be computed from equations as they are needed, or their values may be computed for points on the spiral 10, 20, 30, etc. feet from the P. S_2, and these values tabulated. At the end of this Section, such tables are given, computed for fourteen different spirals. The unit degree of curve in Table II is small, and the spiral turns off very slowly from the tangent; in each table, the unit degree is larger than in the preceding; the spiral of Table XV turns off very rapidly from the tangent.*

*Eleven of these tables are taken by permission from an excellent treatise by Arthur N. Talbot, C. E., entitled "The Railway Transition Spiral." The others have been calculated for this work.

The use of these tables saves a great deal of labor. Sometimes, however, a spiral must be inserted for which no table is at hand, and then the various angles and distances must be computed by such formulas as have been given in the foregoing pages. If an engineer often has to use spirals for which he has no tables, he should compute the tables himself; by so doing he will save considerable time and expense, and otherwise expedite the work.

In the tables here given, the values of δ and θ are tabulated for fourteen different spirals. Above the first column of each table, there is given the value a of the unit degree of curve of spiral for which the table is computed. The first column, headed l, contains the lengths, in feet, of the various arcs CP, Fig. 7, between the P. S_1 and the corresponding points on the spiral. The second column, headed d, gives the degrees of curve of spiral at these points; the third column gives the corresponding deviation angle δ, and the fourth column the deflection angle θ.

EXAMPLE 1.—A spiral 200 feet long connects a tangent with a 5° circular curve. To find the degree of curve, the deviation angle, and the deflection angle to points 40 feet apart on the spiral.

SOLUTION.—By formula **3**, Art. **12**, $a = \frac{D}{L} = \frac{5^\circ}{2} = 2^\circ\ 30'$. This corresponds to Table XII.

For the first point, 40 feet from P. S_1, we find, in the same horizontal line with 40, $d = 1^\circ$, $\delta = 12'$, and $\theta = 4'$. For the second point, we look in the table opposite 80 in the first column, and similarly with the others. The results are as follows:

DISTANCE (FEET) OF POINT FROM P. S_1	DEGREE OF CURVE OF SPIRAL	DEVIATION ANGLE	DEFLECTION ANGLE
40	1° 0′	0° 12′	0° 4′
80	2° 0′	0° 48′	0° 16′
120	3° 0′	1° 48′	0° 36′
160	4° 0′	3° 12′	1° 4′
200	5° 0′	5° 0′	1° 40′

EXAMPLE 2.—The station number of the P. S_1 is 68 + 25. The unit degree of curve of spiral is 1° 40′, and the length of spiral is 400 feet. To find the deflection angles to even stations of the spiral.

SOLUTION.—Since $a = 1^\circ\ 40'$, Table X should be used. Sta. 69 is 75 ft. from the P. S_1; therefore, the deflection angle to Sta. 69 = 8′

$+\frac{5}{10}\times(10.5' - 8') = 9.2'$. The rest of the computation is as follows.

STATION	l, IN FEET		
69	75	$\theta = 8' + \frac{5}{10}\times(10.5' - 8') = 9.2'$	Ans.
70	175	$\theta = 48' + \frac{5}{10}\times(54' - 48') = 51.0'$	
71	275	$\theta = 2°\ 1.5' + \frac{5}{10}\times(2°\ 10.5' - 2°\ 1.5') = 2°\ 6.0'$	
72	375	$\theta = 3°\ 48' + \frac{5}{10}\times(4°\ .5' - 3°\ 48') = 3°\ 54.2'$	
P. S_2	400	$\theta = 4°\ 26.5'$	

EXAMPLES FOR PRACTICE

1. In the following two examples, find the degree of curve of spiral, the deviation angle, and the deflection angle to points on the spiral 30, 60, 90, etc. feet from the P. S_1, and also the values of these angles at the P. S_2: (*a*) Spiral 200 feet long; degree of circular curve = 4°. (*b*) Spiral 160 feet long; degree of circular curve = 2°.

Ans. (*a*) Degrees of curve: 36′, 1° 12′, 1° 48′, 2° 24′, 3°, 3° 36′, and 4°; deviation angles: 5.5′, 21.5′, 48.5′, 1° 26.5′, 2° 15′, 3° 14.5′, and 4°; deflection angles: 2′, 7′, 16′, 29′, 45′, 1° 5′, and 1° 20′
(*b*) Degrees of curve: 22.5′, 45′, 1° 7.5′, 1° 30′, 1° 52.5′, and 2°; deviation angles: 3.5′, 13.5′, 30.5′, 54′, 1° 24.5′, and 1° 36′; deflection angles: 1′, 4.5′, 10′, 18′, 28′, and 32′

2. If the station number of the P. S_1 is 116 + 38, the length of spiral 200 feet, and the unit degree of curve of spiral 3° 20′, find the degree of curve of spiral at each station and at the P. S_2.

Ans. 2° 4′, 5° 24′, and 6° 40′

3. In example 2, find the deflection angles to each station and to the P. S_2. Ans. 0° 12.8′, 1° 27.2′, and 2° 13′

19. Angle Between Chord and Tangent.—In Fig. 7, the exterior angle KNP of the triangle NPC is equal to the sum of the two opposite interior angles; that is,

$$TNP = NCP + CPN$$

or,

$$\delta = \theta + CPN$$

Therefore,

$$CPN = \delta - \theta$$

When running in a spiral, if the transit is moved forwards to some point P, and a backsight taken on P. S_1, then CPN is the angle that must be deflected from this direction to bring the telescope tanget to the spiral at P.

EXAMPLE.—A spiral 300 feet long connects with a 15° curve. When the stake at P. S_2 had been set, the transit was moved forwards to this point, and a backsight taken on P. S_1. Required, the angle that

must be deflected from this direction to bring the telescope tangent to the spiral and circular curve at P. S_2.

SOLUTION.—Here, $a = 15° \div 3 = 5°$ (Art. **12**); therefore, Table XIV must be used. The transit point is 300 ft. from the P. S_1; in the table, opposite 300, we find $\delta = 22° 30'$, and $\theta = 7° 29'$. By the formula in this article, the required angle is $22° 30' - 7° 29' = 15° 1'$. Ans.

20. Coordinates of the Spiral.—Let P, Fig. 8, be any point of a spiral, and PR the perpendicular distance from

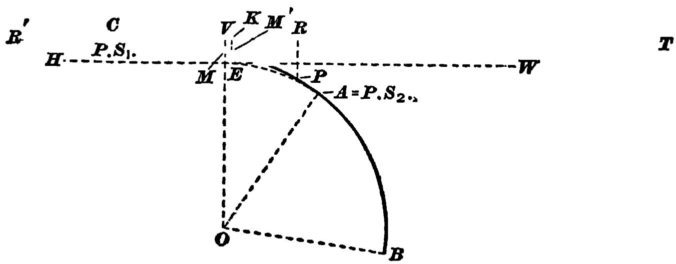

FIG. 8

this point to the original tangent. This perpendicular is represented by y, and its value in feet is given by the formula

$$y = PR = .291\, a\, l^3 - a^3 M \qquad (1)$$

in which a = unit degree of curve of spiral;
l = distance CP from P. S_1 to P, expressed in stations.

The value of M corresponding to any value of l may be taken from the accompanying table:

l	M	l	M	l	M
3.0	.003	5.5	.241	8.0	3.314
3.5	.010	6.0	.442	8.5	5.065
4.0	.026	6.5	.775	9.0	7.557
4.5	.059	7.0	1.301	9.5	11.033
5.0	.124	7.5	2.109	10.0	15.800

The distance CR, measured along the original tangent from the P. S_1 to the foot of the perpendicular PR, is represented by x. This distance is somewhat shorter than the distance CP measured along the curve: the difference in

length between CR and CP is called the x **correction,** and is given by the formula

$$x = CP - CR = .000762\, a^2\, l^5 \qquad (2)$$

This formula gives the quantity to be subtracted from CP, expressed in feet, to obtain the length CR in feet.

The values of y and the x correction, expressed in feet, corresponding to different points of the various spirals, are given in the sixth and seventh columns of Tables II to XV.

EXAMPLE.—To find the values of PR and CR to a point of the spiral 310 feet from the P. S_1 in the example of Art. **17**.

SOLUTION BY USING FORMULAS **1** AND **2**.—We have, in this example, $a = 2°$, $l = 3.1$, and from the small table in this article, using interpolation,

$$M = .003 + \tfrac{1}{5}(.010 - .003) = .004$$

Substituting these values in formula **1**,

$$y = .291 \times 2 \times 3.1^3 - 2^3 \times .004 = 17.31 \text{ ft.} \quad \text{Ans.}$$

Substituting known values in formula **2**,

$$x \text{ cor.} = .000762 \times 2^2 \times 3.1^5 = .9 \text{ ft.}$$

The distance $l = 310$ ft.; therefore, the distance $CR = 310 - .9 = 309.1$ ft. Ans.

SOLUTION BY USING THE TABLES.—From Table XI, in a horizontal line with $l = 310$ of the first column, we find $y = 17.31$ and x cor. $= .9$. Therefore, $PR = 17.31$ ft., and $CR = 310 - .9 = 309.1$ ft., as before.

EXAMPLES FOR PRACTICE

Find, from the tables, the distances PR and CR for the point indicated in each of the following four spirals:

1. $l = 400$ feet; $a = 1° 40'$. Ans. 30.92 ft. and 397.8 ft.
2. $l = 300$ feet; $a = 1°$. Ans. 7.85 ft. and 299.8 ft.
3. $l = 400$ feet; $a = 2°$. Ans. 37.04 ft. and 396.9 ft.
4. $l = 302$ feet; $a = 5°$. Ans. 39.61 ft. and 297.2 ft.

21. The Spiral Offset and the *t* Correction.—Let the circular curve BA, Fig. 8, be produced backwards until at a point E it becomes parallel to the original tangent—that is, until the tangent HW to the circular curve becomes parallel to $R'T$.

The point E at which a spiraled circular curve, if produced backwards, becomes parallel to the original tangent is called **the point of curve,** and is denoted by P. C.

The offset EV from the point of curve to the original tangent is called the **spiral offset.** The spiral offset is a very important distance in laying out the transition spiral. It is represented by F, and its value, in feet, is given by the formula

$$F = .072709\, a L^3 \qquad (1)$$

in which a = unit degree of curve of spiral;

L = whole length of spiral expressed in stations.

If M', Fig. 8, is the middle point of the spiral—that is, a point half way between the P. S_1 and the P. S_2—it will always be found that the spiral offset cuts the spiral at a point M that is a very short distance to the left of M'. The distance CV will therefore always be slightly less than the distance CM'. The difference between the half length of spiral CM' and the distance CV from the P. S_1 to the foot of the spiral offset is called the **t correction;** it is denoted by t, and its value, in feet, is given by the formula

$$t = CM' - CV = .000127\, a^2 L^5 \qquad (2)$$

This correction must be subtracted from the half length of spiral, expressed in feet, to obtain the distance CV, in feet.

The values of F and t are given in the fifth and eighth columns of the tables. The value of l in the first column, corresponding to which we find F and the t correction, is to be taken as the whole length of the spiral.

EXAMPLE.—To find the distances EV and CV for a spiral 400 feet long that connects with a 2° curve.

SOLUTION BY USING FORMULAS 1 AND 2.—Here, $a = \frac{D}{L} = \frac{2°}{4} = \frac{1}{2}$. The whole length of spiral is 4 sta. Therefore, substituting in formula 1,

$$F = .072709 \times \tfrac{1}{2} \times 4^3 = .072709 \times \tfrac{1}{2} \times 64 = 2.33 \text{ ft. Ans.}$$

By formula 2,

$$t \text{ correction} = .000127 \times \tfrac{1}{2}^2 \times 4^5 = .033 \text{ ft.}$$

Therefore,

$$CV = \tfrac{1}{2} \times 400 \text{ ft.} - .033 \text{ ft.} = 199.97 \text{ ft. Ans.}$$

SOLUTION BY USING THE TABLES.—Since $a = \frac{1}{2}°$, Table II should be used. Since the whole length of spiral is 400 ft., we look along the same horizontal line with 400 of the first column, and find F = 2.33; t cor. = .03. Therefore, as before, EV = 2.33 ft., and CV = 199.97 ft. Ans

EXAMPLES FOR PRACTICE

Find the spiral offset and the distance CV for each of the following four spirals:

1. $a = 1°$; length = 400 feet. Ans. 4.65 and 199.87 ft.
2. $a = 1° 40'$; length = 440 feet. Ans. 10.30 and 219.4 ft.
3. $a = 0° 30'$; length = 650 feet. Ans. 9.97 and 324.63 ft.
4. Spiral 500 feet long, connecting with a 4° curve. Ans. 7.26 and 249.75 ft.

22. The Middle Point of the Spiral Offset.—If M', Fig. 8, is the middle point of the spiral, and $M'K$ is its offset from the original tangent, it will be found that, even in the longest spirals, $M'K$ is almost exactly equal to one-half the spiral offset VE. The distance CK from the P. S_1 to the foot of $M'K$ is almost exactly equal to the distance CV from the P. S_1 to the foot of the spiral offset. Consequently,

The spiral offset and the spiral very nearly bisect each other; the point M at which the spiral cuts the offset is almost exactly half way between the P. C. and the original tangent.

A knowledge of this fact is of much use in the selection of spirals to fulfil given topographical conditions, as will be explained further on.

EXAMPLE.—If the unit degree of curve of spiral is 1° 15′, and the length of spiral is 500 feet, find the spiral offset VE, the offset KM' to the middle point of spiral, and the distances CV and CK from P. S_1 to the foot of these offsets.

SOLUTION.—From Table IX, opposite the length 500 in the first column, we find $F = 11.33$ ft. Ans.

Since $CM' = 250$ ft., we find from the table, opposite 250 in the first column, $y = M'K = 5.67$ ft. Ans.

Thus, $M'K$ is almost exactly equal to $\frac{1}{2}\ VE$. Similarly,

$$\left.\begin{aligned} CV &= 250 - .6 = 249.4 \text{ ft.} \\ CK &= 250 - .1 = 249.9 \text{ ft.} \end{aligned}\right\} \text{Ans.}$$

TANGENT DISTANCES

23. Problem I.—*Given a spiraled circular curve, to find the distance from the point of intersection of the tangents to each P. S_1, when the lengths of the two spirals are equal.*

Let $R\,T$ and $R'\,T$, Fig. 9, be the two tangents intersecting in T. Let $A\,B$ be the circular curve, and $C\,A$ and $C'\,B$ the two equal spirals. It is desired to find the distance $T\,C = T\,C'$.

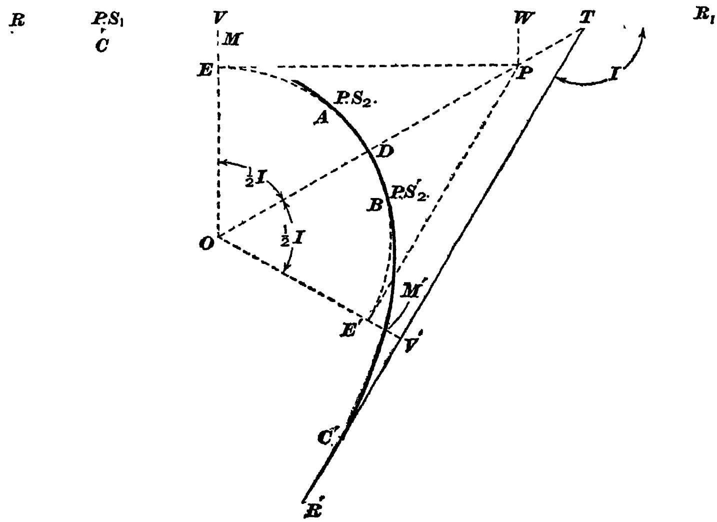

FIG. 9

Let O be the center of the arc $A\,B$. Draw $O\,V$ and $O\,V'$, perpendicular, respectively, to $R\,T$ and $R'\,T$, and produce $A\,B$ until it meets these lines in E and E'. These points are called, respectively, the point of curve P. C. (Art. 21), and the **point of tangent** P. T. of the produced circular arc $A\,B$.

Draw $O\,T$. In the right triangles $V\,O\,T$ and $V'\,O\,T$, $O\,V = O\,V' = R + F$, and $O\,T$ is common to both triangles; therefore, the triangles are equal, and the angle $V\,O\,T = V'\,O\,T = \frac{1}{2}\,I$. Hence,

$$V\,T = V'T = O\,V \tan \tfrac{1}{2} I = (R + F) \tan \tfrac{1}{2} I$$

The desired distance is CT, which equals $VT + CV$; or, substituting for VT the value just found, and for CV the value ($\frac{1}{2}$ length of spiral $-$ t correction) (Art. **21**),

$$CT = C'T = \tfrac{1}{2} \text{ length} - t \text{ cor.} + (R + F) \tan \tfrac{1}{2} I$$

EXAMPLE.—A 6° curve is connected at both ends with the tangents by equal spirals, each 300 feet long. To find the distance from the point of intersection of the tangents to P. S_1 of each spiral, if the angle between the tangents is 80° 20′.

SOLUTION.—Here $a = 6 \div 3 = 2°$, and, therefore, Table XI should be used. In this table, opposite 300 in the first column, we find $F = 3.91$ ft., and t cor. $= .1$ ft.

The radius R of the circular curve is $5{,}730 \div 6 = 955$ ft.; the half length of spiral is $\frac{1}{2} \times 300 = 150$ ft. Substituting these values in the foregoing formula,

$$CT = C'T = 150 - .1 + (955 + 3.91) \tan 40° 10' = 959.3 \text{ ft. Ans.}$$

24. Problem II.—*Given a spiraled circular curve, to find the distance from the point of intersection of the tangents to each P. S., when the lengths of the two spirals are not equal.*

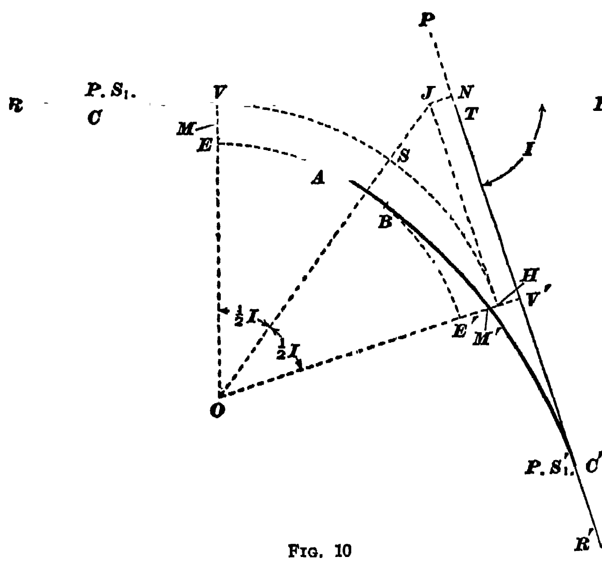

FIG. 10

If, Fig. 10, with a radius OV and a center O, a circular arc VSH is drawn, we shall have, exactly as in the last article,

$$VJ = JH = OV \tan \tfrac{1}{2} I = (R + F) \tan \tfrac{1}{2} I$$

If, from J, a perpendicular JN to the tangent $R'T$ is drawn, then $JN = HV' = E'V' - EV$. But $E'V'$ is the spiral offset to the second spiral, and EV is the spiral offset to the first spiral. Writing $E'V' = F'$ and $EV = F$, we have, therefore, $JN = F' - F$.

(*a*) *To Find the Distance CT.*—The figure gives

$$CT = CV + VJ + JT \qquad (1)$$

Now, $CV = \frac{1}{2} \times$ length $CA - t$ cor. (see Art. **21**); also, $VJ = (R + F) \tan \frac{1}{2} I$; and, from the small triangle JTN,

$$JT = \frac{JN}{\sin JTN} = \frac{JN}{\sin I} = \frac{F' - F}{\sin I}$$

Therefore, substituting these values in (1),

$$CT = \tfrac{1}{2} \text{ length of spiral} - t \text{ cor.}$$
$$+ (R + F) \tan \tfrac{1}{2} I + \frac{F' - F}{\sin I} \qquad \textbf{(1)}$$

This formula gives the distance from the P. S_1 of the shorter spiral to the point of intersection of the tangents.

(*b*) *To Find the Distance $C'T$.*—The figure gives

$$C'T = C'V' + V'N - TN \quad (2)$$

Now, $C'V' = \frac{1}{2} \times$ length of second spiral $- t$ cor. for this spiral;

$$V'N = JH = (R + F) \tan \tfrac{1}{2} I;$$

and, in the triangle JTN,

$$TN = JN \cot JTN = (F' - F) \cot I$$

Therefore, substituting these values in (2),

$$C'T = \tfrac{1}{2} \text{ length of spiral} - t' \text{ cor.}$$
$$+ (R + F) \tan \tfrac{1}{2} I - (F' - F) \cot I \quad \textbf{(2)}$$

This formula gives the distance from the P. S_1 of the longer spiral to the point of intersection of the tangents.

Example.—Two tangents intersect at Sta. 820, and are to be connected with a 5° circular curve. The length of the first spiral is to be 250 feet, and that of the second is to be 400 feet; the angle between the tangents is 31° 48′. To find the station number of P. S_1, and the distance from the point of intersection of the tangents to P. S_1'.

Solution.—The unit degree of curve of the first spiral is 5° ÷ 2.5 = 2°.
From Table XI, first spiral offset = F = 2.27 ft.
The unit degree of curve of the second spiral = 5° ÷ 4 = 1.25°.
From Table IX, second spiral offset = F' = 5.8 ft.

(*a*) *To find the tangent distance CT for the shorter spiral.*—Here, $R = 5{,}730 \div 5 = 1{,}146$ ft.; $F = 2.3$ ft.; and $I = 31° 48'$.

Therefore, $(R + F) \tan \frac{1}{2} I = (1{,}146 + 2.3) \tan 15° 54'$. = 327.1 ft.
Also, $(F' - F) \div \sin I = (5.8 - 2.27) \div \sin 31° 48'$ . . . = 6.7 ft.
And t cor. (Table XI) = .0; $\frac{1}{2}$ length − t cor. = 125 − .0 = 125.0

Therefore, the desired distance CT . . . . = 458.8 ft. Ans.

The station number of P. S_1 will be 820 − (4 + 58.8) = 815 + 41.2. Ans.

(*b*) *To find the tangent distance $C'T$ for the longer spiral.*—
As before, $(R + F) \tan \frac{1}{2} I$ . . . . = 327.1 ft.
t cor. (Table IX) = .2; $\frac{1}{2}$ length − t cor. = 200 − .2 . . = 199.8

Sum = 526.9 ft.
$(F' - F) \cot I = (5.8 - 2.27) \cot 31° 48'$ . . . . = 5.7

Therefore, $C'T$ . . . . = 521.2 ft. Ans.

To find the two points of spiral in the field, it would therefore be only necessary to run the two tangents to their point of intersection, and then to measure back on the first tangent the distance 458.8 ft., and on the second tangent 521.2 ft.

EXAMPLES FOR PRACTICE

1. If a circular curve is connected with the two tangents by spirals of equal lengths, find the distance from the point of intersection of the tangents to the P. S_1, the data being as follows: length of spiral = 800 feet; degree of circular curve = 4°; angle between tangents = 65°. Ans. 1,323.4 ft.

2. Two tangents that intersect at an angle of 50° are to be connected with a 5° circular curve by spirals. The first spiral is to be 400 feet long and the second 500 feet. Find the distances from the point of intersection of the tangents to the P. S_1 of each spiral.
Ans. For 400-ft. spiral, 741.20 ft.; for 500-ft. spiral, 783.96 ft.

TO LAY OUT A SPIRAL IN THE FIELD

I. WHEN THE P. S_1 OF EACH SPIRAL IS VISIBLE FROM THE P. S_2

25. First Method.—Let RT and $R'T$, Fig. 11, be the two tangents that are to be connected with the circular curve AB by the two spirals CA and $C'B$. It will be assumed that the two spirals are of equal length.

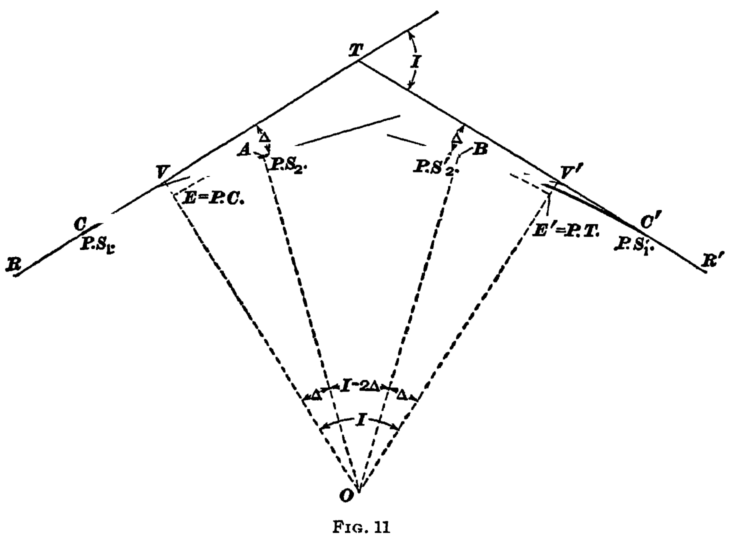

Fig. 11

Compute the unit degree of curve of spiral (Art. **12**); the spiral offset $VE = V'E'$ (Art. **21**); and the distance $CV = C'V'$ (Art. **21**); or obtain these quantities with the help of the tables, and compute the distance $CT = C'T$ (Art. **23**). Run the two tangents to their point of intersection T, measure back from T the distances TC and TC', and at C and C' set stakes marked P. S_1.

Set up the transit at P. S_1, sight on T, and then set stakes on the spiral exactly as on a simple circular curve, except that the deflection angle for each stake is computed by the formula in Art. **17**, or taken from the tables. When the stake at A (marked P. S_2) has been set, move the transit to A, backsight on P. S_1, and deflect from this direction the angle necessary to bring the telescope tangent to the simple circular curve at A. This angle is computed as explained in Art. **16**. Run in the circular curve as usual.

When the stake at B (marked P. S_2') has been set, move the transit to C', backsight on T, and stake out the second spiral in exactly the same manner as the first, using the deflection angles computed for the first spiral. When the last stake of $C' B$ has been set, backsight on T, and continue the survey along the tangent $T R'$.

EXAMPLE —Two tangents that intersect at an angle of 80° 20′ are to be connected with a 6° circular curve by two equal spirals, each 300 feet long. The tangents intersect at Sta. 36. To lay out the two spirals and the circular curve.

SOLUTION.—(*a*) THE COMPUTATIONS.—In the example of Art. **23**, the following values were found for this curve:

Unit degree of curve of spiral = 2°; spiral offset = 3.91 ft.; $C V = C' V' = \frac{1}{2}$ length $- t$ cor. = 149.9 ft.; and $C T = C' T$ = 959.3 ft.

Since T is at Sta. 36, the station number of the P. S_1 is

$$36 - (9 + 59.3) = 26 + 40.7$$

It will be assumed that stakes are set 50 ft. apart on the spirals, and at the even stations on the circular curve.

Transit at P. S_1.—From Table XI, the following deflection angles are found:

to first stake,	0° 5′	(A) Angles to be deflected from the tangent. Vernier set at 0° 0′.
to second stake,	0° 20′	
to third stake,	0° 45′	
to fourth stake,	1° 20′	
to fifth stake,	2° 5′	
to P. S_2 at 29 + 40.7,	3° 0′	

From Table XI, deviation angle to P. $S_2 = \Delta = A O V$, Fig. 11, = 9° 0′. Therefore, central angle of circular curve is

$$80° 20' - 2 \times 9° = 62° 20'$$

The length of $A B$ is therefore 62° 20′ ÷ 6 = 10.389 sta., and the station number of B is

$$29 + 40.7 + (10 + 38.9) = 39 + 79.6$$

By Art. **19**, the angle between the chord CA and the tangent to the circular curve at A is $\Delta - \theta = 9° - 3° = 6°$.

Transit at P. S_2.—The deflection angles to the stakes on the circular curve are as follows:

to Sta. 30, $.593 \times 3°$	= 1° 47′;	to Sta. 35, 16° 47′	(B) Angles to be deflected from tangent to circular curve. Vernier set at 6° 0′.
to Sta. 31,	4° 47′;	to Sta. 36, 19° 47′	
to Sta. 32,	7° 47′;	to Sta. 37, 22° 47′	
to Sta. 33,	10° 47′;	to Sta. 38, 25° 47′	
to Sta. 34,	13° 47′;	to Sta. 39, 28° 47′	
		to B, 31° 10′	

Transit at P. S_1'.—The angles to be deflected are the same as at P. S_1. The station number of P. S_1' is $(39 + 79.6) + 3 = 42 + 79.6$.

(*b*) THE FIELD WORK.—Run the two tangents to their intersection. Measure back from T the distances $TC = TC' = 959.3$ ft., and set stakes marked P. S_1 at C and C'. Set the transit at C with the vernier at 0° 0′; sight on T and deflect the angles (A) to locate the first spiral. When the stake at A (marked P. S_2) has been set, move to this point, set the vernier at 6° 0′, backsight on C, turn the telescope until the vernier reads 0° 0′, and from this direction deflect the angles (B) to locate the circular curve. When the stake at B (marked P. S_2) has been set, move the transit to C', set the vernier at 0° 0′, backsight on T, and deflect the angles (A) to locate the second spiral.

26. Second Method.—Another method consists in setting the transit over the point E, Fig. 11 (see Art. **21**), running in the entire circular curve from the P. C. to the P. T., setting stakes at each P. S_1 and P. S_2, and locating the spirals afterwards. The details of this method are as follows:

Run the tangents to their intersection, measure back the distances TC and TC' to locate each P. S_1, and set stakes marked P. S_1 at these points. Measure forwards the distances CV and $C'V'$, offset the distances VE and $V'E'$, and set stakes marked P. C. and P. T. at E and E', respectively.

Set the transit over E, bring the telescope parallel to RT, and run in the complete circular curve EE', setting stakes (marked P. S_2) at A and B, and on the curve between A and B. The angle $EOA = BOE' =$ the total deviation angle Δ, and therefore the angle of the circular curve AB is $I - 2\Delta$. Therefore, the distance EA is $\frac{\Delta}{D}$ stations; AB is $\frac{I - 2\Delta}{D}$ stations; and BE' is $\frac{\Delta}{D}$ stations.

The P. S_1 and P. S_2 of each spiral having been thus located, each curve is run in with the transit at its P. S_1, as described in Art. 25.

27. Spirals of Unequal Lengths.—Where a circular curve is to be spiraled at both ends, the two spirals should in all cases be chosen of equal lengths where this is possible. The unit degree of curve of spiral, the tangent distances CT and $C'T$, and all of the deviation and deflection angles will then be the same for both spirals, and but a single computation will be necessary for both curves. Sometimes, however, this selection is impossible. Generally, the longest spirals will give the easiest riding; but, where curves are numerous, in order to prevent overlapping, a spiral much shorter than is theoretically desirable must sometimes be chosen for one end of the circular curve.

The method of laying out two unequal spirals is the same in principle as that described for equal spirals, except that the computations must be made separately for each spiral.

II. WHEN THE P. S_1 IS NOT VISIBLE FROM THE P. S_2

28. Corresponding Points.—If P is any point of a spiral, then any two points whose distances from P are equal and which lie, one on the spiral and the other on the osculating circle at P, are called **corresponding points.**

Let P, Fig. 12, be any point of a spiral, and let $PM'N$ be the osculating circle to the spiral at this point. Let P' be a second point on the spiral, and P'' a point on the circle at the same distance from P, so that arc PP'' = arc PP'. Then, P' and P'' are corresponding points. Similarly, P_1' and P_1'' are corresponding points if arc PP_1' = arc PP_1''.

It is shown by advanced mathematics that the angle $P'PP''$ is equal to the deflection angle TCP''' to a point P''' on the spiral whose distance CP''' from C is equal to the distance PP' or PP''. Therefore, the angle $P'PP''$ can be readily computed or taken from the tables.

To find the angle between the tangent KPM and the chord PP', we have,

$$MPP' = MPP'' + P''PP'$$

But MPP'' is the deflection angle from the tangent to the circular curve $M'PN$; and, since $P''PP' = TCP'''$, we have the following rule for finding the angle that must be deflected from the tangent to the spiral at P to locate any second point P' of the spiral:

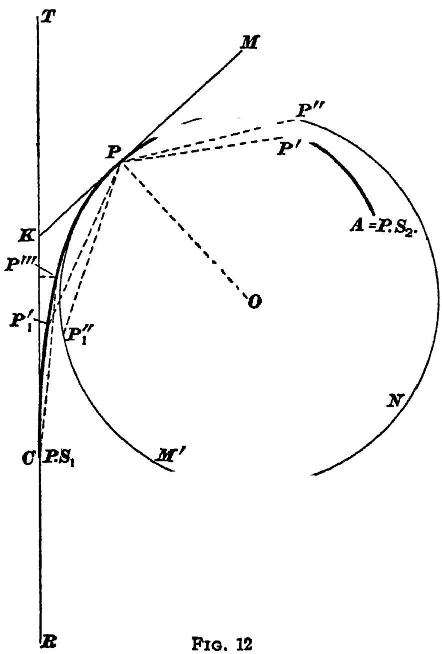

FIG. 12

Rule.—*Find the degree of curve of spiral at the point P (Art.* **12***); this is the same as the degree of curve of the circular arc PN of the osculating circle. Find the angle that would be deflected to locate the corresponding point on the osculating circle, and also the spiral deflection angle TCP''' to a point on the spiral whose distance from the P. S_1 is equal to PP' or PP''. If the point P' is between P and the P. S_2, add the last angle to the first; otherwise, subtract the last angle from the first: the result is the angle desired.*

The last part of the rule is evident from Fig. 12, since $KPP_1' = KPP_1'' - P_1'PP_1'' = KPP_1'' - TCP'''$, if $PP_1' = CP'''$.

EXAMPLE.—A spiral 600 feet long connects a tangent with a 6° curve. The P. S_2 is not visible from the P. S_1, so that the transit is moved forwards on the spiral to a point 300 feet from the P. S_1, and the telescope is brought tangent to the spiral at this point. It is required to compute the deflection angles necessary to locate this spiral, the stakes being set 100 feet apart.

SOLUTION.—The unit degree of curve of spiral is $6° \div 6 = 1°$. Therefore, Table VII, must be used.

The transit at P. S_1.—The deflections from the original tangent to locate stakes 1, 2, and 3 sta. from the P. S_1 are, from Table VII:

to first stake,	0° 10′	Deflections from original tangent
to second stake,	0° 40′	
to third stake,	1° 30′	

The transit 3 sta. from the P. S_1.—The degree of curve of spiral at this point, from Table VII, is 3° 0′. The stake 400 ft. from P. S_1 is 100 ft. from the transit point. The deflection angle to a point 100 ft. from the transit point on a simple circular curve is $\frac{1}{2} \times 1 \times 3° = 1° 30'$. The deflection angle from the P. S_1 to a point 100 ft. distant on the spiral. by Table VII, is 0° 10′. Since the point is between the transit point and the P. S_2, these two angles must be added, the result being 1° 40′. The complete results are as follows:

STATION	DISTANCE FROM TRANSIT POINT	DEFLECTION ANGLE TO 3° CIRCULAR CURVE	SPIRAL DEFLECTION ANGLE	TOTAL DEFLECTION ANGLE	
4	100	$\frac{1}{2} \times 1 \times 3° = 1° 30'$	0° 10′	1° 40′	Ans.
5	200	$\frac{1}{2} \times 2 \times 3° = 3° \ 0'$	0° 40′	3° 40′	
P. S_2	300	$\frac{1}{2} \times 3 \times 3° = 4° 30'$	1° 30′	6° 0′	

29. The principles of the preceding article enable one to compute the deflection angles necessary to locate either spiral with the transit at its P. S_2. When the telescope has been brought tangent to the spiral and the circular curve at this point, the angle KPP_1', Fig. 12, to be deflected to any stake is $KPP_1'' - P_1'PP_1''$ = (deflection angle to a point on the circular curve) − (spiral deflection angle).

In practice, however, it is better to run in the circular curve first, and then to locate each spiral from its P. S_1.

30. The Field Work.—Where the P. S_2 is not visible from the P. S_1, the complete circular curve should be run in from the P. C. to the P. T., as described in Art. **26**, and the spirals located afterwards. The only change in the field work is that introduced by the necessity of moving the transit forwards on the spiral.

The transit having been set up at some point P, Fig. 12, a backsight is taken on C, and the angle $KPC = \delta - \theta$ is deflected to bring the telescope tangent to the spiral (Art. **19**). Corresponding to the distance from P to each

stake between P and A, the circular deflection angles MPP'' are computed and the spiral deflection angles $P'PP'' = TCP'''$ added to them. The respective sums will be the angles MPP' that must be deflected from the tangent KM to locate the remaining stakes of the spiral.

EXAMPLE.—It is required to run in a spiral 800 feet long connecting with a 4° curve, the transit having been moved forwards to a stake 200 feet from the P. S_1. The stakes are to be set 100 feet apart.

SOLUTION.—Here $a = 4° \div 8 = \frac{1}{2}°$, and Table II should be used.

Transit at P. S_1.—From Table II,

deflection angle to first stake = 0° 5′
deflection angle to second stake = 0° 20′

Transit 200 ft. from P. S_1.—From Table II, $\delta = 1° 0'$ and $\theta = 20'$; therefore, the angle $KPC = 1° - 20' = 40'$. Also, d = degree of osculating circle = 1°.

The vernier is set at 0° 40′, a backsight is taken on C, and the telescope is turned until the vernier reads 0° 0′. From this direction, the following angles are deflected:

Stake	Distance From Transit Point	Deflections to a 1° Circular Curve	Spiral Deflections	Total Deflections
3	100	0° 30′	0° 5′	0° 35′
4	200	1° 0′	0° 20′	1° 20′
5	300	1° 30′	0° 45′	2° 15′
6	400	2° 0′	1° 20′	3° 20′
7	500	2° 30′	2° 5′	4° 35′
P. S_2	600	3° 0′	3° 0′	6° 0′

EXAMPLES FOR PRACTICE

In the following examples, the transit was moved forwards on the spiral. Find the angle KPC, Fig. 12, that must be deflected from the chord to the P. S_1 to bring the telescope tangent to the spiral, and also the angles that must be deflected from this direction to locate stakes 50 feet apart on the spiral between the transit point P and the P. S_2.

1. $D = 10°$; $L = 300$ feet, transit point 250 feet from P. S_1.

Ans. $\begin{cases} KPC = 6° 57' \\ \text{Deflection angles to P. } S_2 = 4° 13' \end{cases}$

2. $D = 10°$; $L = 600$ feet, transit point 400 feet from P. S_1.

Ans. $\begin{cases} KPC = 8° 53.5' \\ \text{Deflection angles, } 1° 44', 3° 36.5', 5° 37.5', \text{ and } 7° 46.5' \end{cases}$

3. $D = 15°$; $L = 300$ feet, transit point 150 feet from P. S_1.

Ans. $\begin{cases} KPC = 3° 45' \\ \text{Deflection angles, } 2° 4.5', 4° 35', \text{ and } 7° 29.5'. \end{cases}$

INSERTING SPIRALS IN OLD TRACK

31. When it is desired to insert transition spirals in track that is already laid, the problem is essentially as follows:

There will be found on the roadbed two tangents $R\,V_1$ and $V_1'\,R'$, Fig. 13, that are connected by a simple circular curve $V_1\,M_1\,V_1'$. This curve is to be replaced by a new curve, either by offsetting the whole curve without increasing its degree of curve, as shown in Fig. 13, or else by sharpening

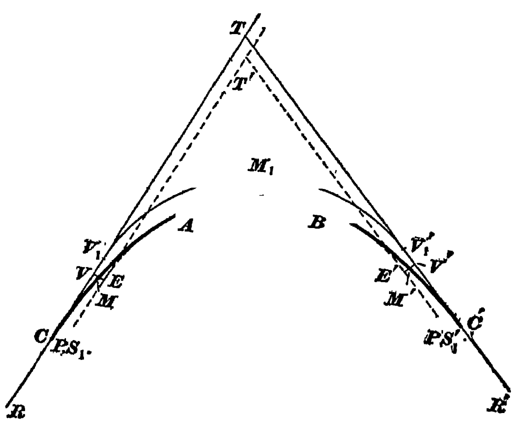

FIG. 13

the old curve, as shown in Figs. 14 and 15, so that the tangents $E\,T'$ and $E'\,T'$, Fig. 13, to the new curve may be offset from the old tangents, and thus give room for the introduction of transition spirals $C\,M\,A$ and $C'\,M'\,B$.

If the degree of the new circular curve $E\,A\,B\,E'$ is made the same as the degree of the old curve $V_1\,M_1\,V_1'$, the effect will be the same as moving the whole curve $V_1\,M_1\,V_1'$ downwards until it is brought tangent to the lines $E\,T'$ and $E'\,T'$ at the points E and E'. The entire length of the old track will thus be changed from its old position, and nearly all of

it will be moved by the whole amount of the spiral offset VE or $V'E'$. This method of replacing the old curve is, therefore, not usually advisable unless the offsets are small or the curve very short. In nearly every case, it will be better to sharpen the old curve in such a manner that, while the track is moved as little as possible on the roadbed, the new tangents ET' and $E'T'$ will still fall a sufficient distance inside of the old ones.

The distances VE and VE' from the old to the new tangents are the same as the spiral offsets of the two spirals. They depend for their value on the degree of the new circular curve and on the lengths chosen for the spirals (Art. **21**). The best length to be taken for a spiral will be determined by the speed of the trains that pass over the track and by the degree of the new circular curve; but this theoretically best length must often be modified by topographical conditions. In any particular case, the engineer has, therefore, to select the spirals and the new degree of the circular curve in such a manner that these quantities will approach as nearly as possible the values that are theoretically the best, and yet so that the old track will not be displaced more than is advisable on the roadbed. This subject will be treated more fully further on.

32. The successive steps in inserting spirals in old track are as follows:

1. Ascertain the degree of curve of the old circular curve $V_1 M_1 V_1'$, Fig. 13.

2. Select the spirals to be inserted and the new degree of circular curve; compute the spiral offsets VE and $V'E'$ (Art. **21**), the distances VC and $V'C'$ (Art. **22**), and the distances TC and TC' (Arts. **23** and **24**).

3. Locate the P. S.$_1$ and run in each spiral as already explained. If the point T is inaccessible, the P. S.$_1$ may be located by measuring back from the old P. C. the distance $V_1 C$, which equals $TC - TV_1$, in which TC is computed as in Arts. **23** and **24**, and TV_1 is the tangent distance to the old (unspiraled) curve.

33. External to a Circular Curve.—The **external distance**, or, for shortness, the **external**, of a circular curve, is the distance, measured along a radial line, between the point of intersection of the tangents and the curve. Thus, in Fig. 9, the external to the circular curve EDE' is PD. The figure gives, denoting the radius by R, and the external by q,

$$q = PD = OP - OD = OE \sec \tfrac{1}{2} I - R = R \sec \tfrac{1}{2} I - R;$$

or, finally,

$$q = R(\sec \tfrac{1}{2} I - 1) = \frac{2R \sin^2 \tfrac{1}{4} I}{\cos \tfrac{1}{2} I} \qquad (1)$$

The difference between the secant of an angle and 1 is called the **external secant** of the angle. Some field books give tables of both natural and logarithmic external secants. The abbreviation *exsec*, read *exsecant*, is used for external secant. Formula **1** may, therefore, be written

$$q = R \operatorname{exsec} \tfrac{1}{2} I \qquad (2)$$

If no table of exsecants is available, $\sec \tfrac{1}{2} I - 1$, or its equivalent $\dfrac{2 \sin^2 \tfrac{1}{4} I}{\cos \tfrac{1}{2} I}$ (the latter being preferable for logarithmic work), may be used instead of exsec I. Tables are also given in some books, from which external distances (values of q) can be taken directly.

34. External to a Spiraled Circular Curve, the Spirals Being Equal.—The external to a spiraled circular curve is the distance TD, Fig. 9, between the curve and the intersection of the tangents to the spirals. Denoting the external by q_1, the figure gives

$$q_1 = PD + PT = q + PW \sec WPT$$
$$= q + PW \sec \tfrac{1}{2} I = q + EV \sec \tfrac{1}{2} I;$$

or, since $EV = F$ (Art. **21**),

$$q_1 = q + F \sec \tfrac{1}{2} I \qquad (1)$$

If a table of externals for circular curves is available, q may be taken from it. If not, the value of q_1 may be written, by replacing the value of q from formula **1**, Art. **33**,

$$q_1 = R(\sec \tfrac{1}{2} I - 1) + F \sec \tfrac{1}{2} I;$$

or

$$q_1 = (R + F) \sec \tfrac{1}{2} I - R \qquad (2)$$

EXAMPLE.—A 6° curve is connected at both ends with the tangents by spirals 300 feet long. To find the external to the spiraled curve, if the angle between the tangents is 80° 20′.

SOLUTION.—From the table of radii and deflections in *Circular Curves*, the radius of a 6° curve is found to be 955.37 ft.

The unit degree of curve of spiral is 6° ÷ 3 = 2°; therefore, from Table XI, $F = 3.91$ ft. Substituting these values in formula **2**,

$$q_1 = (955.37 + 3.91) \sec 40° 10' - 955.37 = 299.9 \text{ ft. Ans.}$$

35. Problem.—*To find the length of the transition spiral when the degree of the circular curve and the spiral offset are given.*

From formula **1**, Art. **21**, we have $F = .072709\, a L^3$, and from formula **3**, Art. **12**, $a = \frac{D}{L}$. Substituting this value of a in the first formula,

$$F = .072709\, L^2 D \qquad (1)$$

Solving for L,

$$L = \sqrt{\frac{F}{.072709\, D}} = 3.7086 \sqrt{\frac{F}{D}} \qquad (2)$$

EXAMPLE.—What length of spiral will give a spiral offset of 11 feet, if the degree of the circular curve is 7° 30′?

SOLUTION.—Substituting $F = 11$ and $D = 7.5$ in formula **2**,

$$L = 3.7086 \times \sqrt{\frac{11}{7.5}} = 4.49 \text{ sta.} = 449 \text{ ft. Ans.}$$

A spiral 450 ft. long would be chosen. The throw of the old track at the P. C. (Fig. 8) is the distance VM, which would then be about $5\frac{1}{2}$ ft. (Art. **22**).

EXAMPLES FOR PRACTICE

1. A 5° curve is connected at both ends with the tangents by spirals 400 feet long. What is the external to the spiraled curve, if the angle between the tangents is 75° 36′? Ans. 311.9 ft.

2. Find the length of spiral that will produce the spiral offset indicated when the spiral joins each of the following curves with the tangent. (*a*) $D = 2° 24'$; $F = 1$ foot; (*b*) $D = 4° 6'$; $F = 5$ feet; (*c*) $D = 3° 30'$; $F = 2$ feet; (*d*) $D = 8° 30'$; $F = 16$ feet.

Ans. (*a*) 239 ft.; (*b*) 410 ft.; (*c*) 280 ft.; (*d*) 509 ft.

SELECTION OF SPIRALS

36. The Best Length of Spiral.—Equation (1), Art. **13**, is $e = .000058\, a\, L\, V^2$. This equation gives the total superelevation attained by the outer wheels in passing over the whole spiral. The best length of spiral should be such that, for all spirals and their corresponding train speeds, the time occupied by the outer wheels in attaining a given superelevation shall be constant; that is, the superelevation attained by the outer wheels during, for example, 1 second should be the same for all curves and spirals.

Let T be the time, in hours, occupied by the train in passing over the whole spiral; then

$$T = \frac{L}{52.80\, V}$$

since there are 52.80 stations in 1 mile. The time T is the number of hours required by the outer wheels to attain a superelevation e. The superelevation attained in one unit of time will therefore be

$$\frac{e}{T} = .000058\, a\, L\, V^2 \div \frac{L}{52.80\, V} = .000058 \times 52.80 \times a\, V^3$$

Since this superelevation attained in a unit of time is to be a constant for all spirals, $a\, V^3$ must be a constant. We may therefore write

$$a\, V^3 = K$$

in which K is some constant quantity whose value is to be determined.

It is found from experience that, when $V = 60$ miles per hour, the best spiral is probably that in which $a = \frac{1}{2}°$. Therefore, substituting $\frac{1}{2}$ for a and 60 for V, we obtain $K = \frac{1}{2} \times (60)^3$. By finally replacing this value of K, and solving for a, we find

$$a = \frac{1}{2} \times \left(\frac{60}{V}\right)^3$$

This formula gives the value of a for the easiest riding when the maximum train velocity V is known. If this value of a is substituted in the formula $L = \frac{D}{a}$ (Art. **12**), the resulting value of L will be the best length of spiral.

EXAMPLE.—To find the theoretically best length of spiral to connect with a 6° curve, the maximum train velocity being 40 miles per hour.

SOLUTION.—Substituting $V = 40$ in the formula for a, we get

$$a = \tfrac{1}{2} \times (\tfrac{60}{40})^3 = \tfrac{1}{2} \times \tfrac{27}{8} = \tfrac{27}{16}^\circ$$

Therefore, $L = 6 \div \frac{27}{16} = 3.5556$ sta. $= 355.6$ ft. Ans.

37. Table of Minimum Spiral Lengths.—The accompanying table, from Talbot's "Transition Spiral," gives the values of a corresponding to the *least length of spiral* that the engineer should endeavor to insert. The spiral may be longer than the length obtained from this table, but it should not be shorter, unless topographical conditions make it necessary to use a shorter spiral than the minimum given in the table.

Maximum Train Speed Miles per Hour	Unit Degree of Curve of Spiral
75	30′ or less
60	30′ or less
50	1° or less
40	2° or less
30	3° 20′ or less
25	5° or less
20	10° or less

The least length corresponding to any value of a is found from the formula $L = \dfrac{D}{a}$.

EXAMPLE.—To find the least length for the spiral in the example of the preceding article.

SOLUTION.—The velocity is 40 mi. per hr.; therefore, from the table, $a = 2°$, and $L = 6° \div 2 = 3$ sta. $= 300$ ft. Ans.

EXAMPLES FOR PRACTICE

Find the best length of spiral and also the least length that should be selected in each of the following examples:

1. $V = 75$; $D = 2°$. Ans. 781 and 400 ft.
2. $V = 40$; $D = 7°$. Ans. 415 and 350 ft.
3. $V = 30$; $D = 10°$. Ans. 250 and 300 ft.

NOTE.—From the answers to example 3, the student will notice that the theoretically best lengths for very sharp curves and low speeds may be less than the least lengths advised by Talbot. The practice of engineers in selecting spirals for such speeds differ widely.

38. Problem.—*To select a spiral and a new circular curve so that the vertex of the old curve shall not be moved on the roadbed.*

Let $V_1 D V_1'$, Fig. 14, be the old curve, and let $C A B C'$ be the new spiraled curve. The vertex D is not to be moved.

Let $q = TD =$ external to old unspiraled curve;

$q' = PD =$ external to new (unspiraled) circular curve;

$q_1 =$ external to new spiraled curve.

Then, $q_1 = TD = q$, since the vertex is not moved Let R

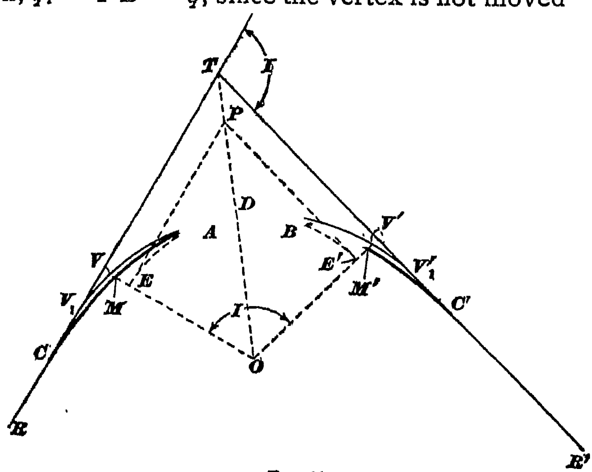

FIG. 14

and R' be the radii of the old and new circular curves, respectively.

From formula **1**, Art. **34**,

$$q_1 = q' + F \sec \tfrac{1}{2} I;$$

or, since $q_1 = q$,

$$q = q' + F \sec \tfrac{1}{2} I;$$

whence

$$q' = q - F \sec \tfrac{1}{2} I \qquad \textbf{(1)}$$

The external q to the old curve can be computed (Art. **33**), taken from a table, or measured on the ground. If a value F is then assumed, formula **1** will give the external q' to the new circular curve. Having q', the radius R' of the new

circular curve is taken from a table of externals or is computed by means of formula **1**, Art. **33**, which gives

$$R' = \frac{q'}{\sec \frac{1}{2} I - 1} = \frac{q'}{\text{exsec} \frac{1}{2} I} \qquad (2)$$

The degree of curve can now be found, and from it and F the value of L is computed by formula **2**, Art. **35**. The spiral may then be laid out on the ground as described in Art. **25**.

39. It will be observed that, in the method of the preceding article, the length of spiral and the new degree of curve were determined entirely by the value chosen for the spiral offset F, or VE. It will very seldom be the case, however, that the amount of this offset must be exactly fixed to satisfy conditions on the ground, though this might be the case if the curve were in a tunnel that could not be widened, or at the entrance to a bridge. A more practical method is as follows:

The degree of the curve $V_1 D V_1'$ having been given, a value is assumed for the length of spiral best adapted to this curve (see Arts. **36** and **37**). Then the unit degree of curve of spiral (Art. **12**), and the spiral offset (Art. **21**) are computed. These quantities cannot be exactly determined, because the spiral does not connect with the curve $V_1 D V_1'$, but with the curve $E D E'$, and the degree of $E D E'$ is not yet known. But if it is assumed that the degree of $E D E'$ is the same as that of $V D V'$, a value of VE will be obtained that will be sufficiently close to the true value to enable the engineer to judge whether the offset that will finally result will be too large or not. If it is inadvisable to throw the track so great a distance as the value of VM corresponding to the length of spiral chosen, a new length of spiral must be assumed, and a new spiral offset computed. Formula **1**, Art. **35**, shows that the smaller L is, the less F will be, and therefore that *shortening the spiral decreases the spiral offset.* Successive smaller values for L must be tried until a length is found that leads to an admissible value of VM. With this value of L, and the corresponding approximate value of F, the new degree of curve is computed as in Art. **38**.

Having L and the degree of curve of $E A B E'$, the value of F is accurately computed, and the curve and spirals are located as described in Art. **31**.

Example.—Two tangents that intersect at an angle of 40° are connected by a 5° circular curve. It is required to select spirals for this curve if the vertex is not to be moved, nor the old track moved at any point more than .7 foot on the roadbed, assuming the maximum train speed to be 50 miles per hour.

Solution.—Since VM, Fig. 14, cannot exceed .7 ft., the spiral offset VE must not be greater than about 1.4 ft. (Art. **22**).

From the formula of Art. **36**, the best length of spiral will be

$$5 \div \tfrac{1}{2} \times (\tfrac{60}{50})^3 = 5 \div \tfrac{216}{250} = 579 \text{ ft.}$$

As the spiral offset must be so small, this spiral would evidently be too long. Spirals shorter than this will be assumed and tested.

First assumption: length = 500 ft. Then, a = 1°, and F (Table VII) = 9.07 ft. A length of 500 ft. is therefore too great.

Second assumption: length = 300 ft. Then, a = 1° 40′, and F (Table X) = 3.26 ft. The spiral is still too long.

Third assumption: length = 200 ft. Then, a = 2° 30′, and F (Table XII) = 1.45 ft. A length of 200 ft. may therefore be taken for the spiral.

Computation of the degree of the spiraled curve.—From a table of radii and deflections, the radius of a 5° curve is found to be 1,146.3 ft. Substituting known values is formula **1**, Art. **33**,

$$q = 1{,}146.3\ (\sec 20° - 1) = 73.478 \text{ ft.}$$

Substituting this value of q in formula **1**, Art. **38**,

$$q' = 73.478 - 1.45 \sec 20° = 71.935 \text{ ft.}$$

Finally, substituting this value of q' in formula **2**, Art. **38**,

$$R' = \frac{71.935}{\sec 20° - 1} = 1{,}122.2 \text{ ft.}$$

The degree of curve corresponding to this radius is 5° 6′. The old curve is thus sharpened slightly. We have, then, a = 5.1° ÷ 2 = 2.55°; and (formula **1**, Art. **21**)

$$F = .072709 \times 2.55 \times 2^3 = 1.48 \text{ ft.}$$

The final selection will therefore be as follows:

$$\left.\begin{array}{ll} \text{Degree of new curve} & = 5° \, 6' \\ \text{Length of spiral} & = 200 \text{ ft.} \\ \text{Spiral offset} & = 1.48 \text{ ft.} \end{array}\right\} \text{Ans.}$$

This is too short a spiral for the train speed given, but it cannot be lengthened without increasing the spiral offset.

40. Problem.—*To select a spiral and the new circular curve so that the old track shall be moved as little as possible on the roadbed.*

The new track must be made to pass as far outside the old track at the vertex as it is offset from the old track at the new P. C. That is, the distance HD, Fig. 15, must be made

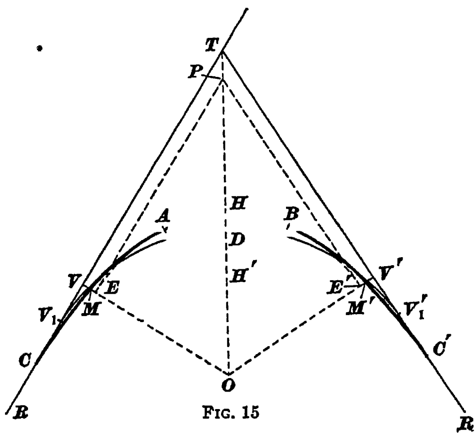

FIG. 15

equal to $VM = \frac{F}{2}$. The new external distance TH must, therefore, equal the old external distance TD diminished by $\frac{F}{2}$. Therefore, if the new external distance is q_1, we have

$$TH = q_1 = TD - \frac{F}{2} = q - \frac{F}{2}$$

But, by formula **1**, Art. **38**,

$$PH = q' = q_1 - F \sec \tfrac{1}{2} I$$

Therefore, $$q' = q - F \sec \tfrac{1}{2} I - \frac{F}{2}$$

In applying this formula, the greatest allowable offset VM, or $\frac{F}{2}$, is first selected, and then the external q to the old curve is computed or taken from a table. The radius and degree of the new curve are found as explained in Art. **38**, the length of spiral by formula **2**, Art. **35**, and the unit degree of curve of spiral by the usual formula $a = \frac{D}{L}$.

EXAMPLE.—In the example of Art. **39**, to select the new degree of curve and the spiral, supposing that the vertex may be moved.

SOLUTION.—The track may be thrown .7 ft. at the new P. C.; therefore (Art. **22**), $F = 1.4$ ft. The external to the old curve was found to be 73.478 ft. Therefore, by the formula in this article,

$$q' = 73.478 - 1.4 \sec 20° - .7 = 73.478 - 1.49 - .7 = 71.288 \text{ ft.}$$

The degree of the new curve, either computed or taken from a table, is 5° 9′, nearly.

The length of spiral (formula **2**, Art. **35**) is

$$3.7086 \times \sqrt{\frac{1.40}{5.15}} = 1.934 \text{ sta.} = 193.4 \text{ ft.}$$

The theoretical selection will therefore be as follows:

$$\left.\begin{array}{ll}\text{Degree of new curve} & = 5° \ 9' \\ \text{Length of spiral} & = 193.4 \text{ ft.} \\ \text{Spiral offset} & = 1.4 \text{ ft.}\end{array}\right\} \text{Ans.}$$

The practical selection would probably be $D = 5° \ 7'$, length $= 192$ ft.; a would then be 2° 40′, and the spiral offset, 1.37 ft.

41. Problem.—*To select a spiral and a new circular curve so that the old track shall be offset a given distance at the new P. C. and also moved a given distance either in or out at the vertex.*

Let VM, Fig. 15, be the distance that the track may be moved in at the new P. C., and let p $(= HD)$ be the distance that it may be moved out at the vertex. Then, $VM = \frac{F}{2}$.

The external PH to the new circular curve will then be (Art. **40**),

$$PH = q - F \sec \tfrac{1}{2} I - p \qquad (1)$$

If the new track is to fall inside of the old track at the vertex, let H' be the position of the new vertex, and let $DH' = p'$. The external PH' to the new curve will then be

$$PH' = q - F \sec \tfrac{1}{2} I + p' \qquad (2)$$

If, therefore, q is computed, and if F and either p or p' is assumed, formulas **1** and **2** will give the value of the new external PH or PH'. From this, the new degree of curve is found, and the solution of the problem is completed exactly as in Arts. **39** and **40.**

EXAMPLE.—A 4° circular curve joins two tangents that intersect at an angle of 30° 20′. It is required to select a new curve and spirals that shall throw the old track out 1 foot at the vertex and .8 foot in at the new P. C.

SOLUTION.—Formula **1** is to be used. The external q to the old 4° curve is found to be 51.72 ft. Also, $F = 2 \times .8 = 1.6$ ft. (Art. **22**), and $p = 1$. Therefore, substituting these values in formula **1**, the external to the new curve is

$$PH = 51.72 - 1.6 \times \sec 15° 10' - 1 = 49.06 \text{ ft.}$$

The degree of the new curve is found, either by computation or from a table of externals, to be 4° 13′.

By formula **2**, Art. **35**,

$$L = 3.7086 \times \sqrt{\frac{1.6}{4.217}} = 2.284 \text{ sta.} = 228.4 \text{ ft.}$$

The theoretical selection is, therefore,

$$\left.\begin{array}{ll} \text{New degree of curve} & = 4° 13' \\ \text{Length of spirals} & = 228.4 \text{ ft.} \\ \text{Spiral offsets} & = 1.6 \text{ ft.} \end{array}\right\} \text{Ans.}$$

In practice, the selection would probably be $D = 4° 13'$, length = 211 ft.; a would then be 2° and Table XI could be employed. The resulting spiral offset would be 1.37 ft.

EXAMPLES FOR PRACTICE

A 3° 30′ curve joins two tangents that intersect at an angle of 26° 20′. Select a new curve and spiral, if the old track may be thrown by the amounts indicated in the following examples (the answers given are the theoretical selections):

1. Thrown out 2 feet at the vertex and in $1\frac{1}{2}$ feet at the new P. C.
Ans. Length = 323.2 ft.; degree of new circular curve = 3° 57′

2. Thrown in $1\frac{1}{2}$ feet at the new P. C.; the vertex must not be moved.
Ans. Length = 330.8 ft.; D = 3° 46′

42. Remarks on the Selection of Spirals.—The student will readily understand that it is impossible to give exact rules for the best selection of spirals. Having located the old P. C. and P. T., and having run over the center line of the old track, the engineer should study the location thus obtained and decide how it may be modified to allow the insertion of transition spirals. He should always try a few different values for the throw at the vertex and at the new P. C., in order to see how the varying of these quantities will affect the new degree of curve and the length of spiral. In many cases he will find that the necessity of keeping the new track on the old roadbed will make it necessary to insert shorter spirals than are theoretically best.

TABLE I

SUPERELEVATION (FEET) OF OUTER RAIL ON CURVES

Degree of Curve	Velocity, in Miles per Hour											
Degrees	10	15	20	25	30	35	40	45	50	60	70	75
1	.006	.013	.023	.036	.051	.070	.091	.116	.143	.206	.283	.325
2	.011	.026	.046	.071	.103	.140	.183	.231	.285	.410	.567	.650
3	.017	.039	.069	.107	.154	.210	.274	.346	.427	.612	.842	
4	.023	.051	.091	.143	.206	.280	.365	.461	.568	.811	1.125	
5	.029	.064	.114	.179	.257	.349	.455	.574	.707	1.006		
6	.034	.077	.137	.214	.308	.418	.545	.687	.844	1.196		
7	.040	.090	.160	.250	.359	.487	.634	.798	.979			
8	.046	.103	.183	.285	.410	.556	.723	.908	1.112			
9	.051	.116	.206	.320	.460	.624	.811	1.017				
10	.057	.129	.228	.356	.511	.692	.898	1.124				
11	.063	.142	.251	.391	.561	.760	.984					
12	.069	.154	.274	.427	.611	.826	1.069					
14	.080	.180	.319	.497	.711	.959						
16	.091	.206	.365	.567	.809	1.088						
18	.102	.231	.410	.637	.906							
20	.114	.256	.455	.707	1.002							
25	.141	.318	.563	.775								
30	.168	.380	.672	.844								
35	.195	.441	.778									
40	.222	.501	.881									
50	.276	.618										

TABLE II

TRANSITION SPIRAL

a = 0° 30′. 1° in 200 feet

l	d	δ	θ	F	y	x cor.	t cor.
	° ′	° ′	° ′	Feet	Feet	Feet	Feet
25	0 7.5	0 0.9	0 0.3	.00	.00	.00	.00
50	15	3.8	1.3	.00	.02	.00	.00
75	22.5	8.4	2.8	.02	.06	.00	.00
100	30	15	5	04	.15	.00	.00
125	0 37.5	0 23.4	0 7.8	.07	.29	.00	.00
150	45	33.8	11.3	12	.49	.00	.00
175	52.5	45.9	15.3	.20	.78	.00	.00
200	1 00	1 00	20	.29	1.16	.01	.00
225	1 7.5	1 15.9	0 25.3	.41	1.66	.01	.00
250	15	33.8	31.3	.57	2.27	.02	.00
275	22.5	53.4	37.8	.76	3.03	.03	.01
300	30	2 15	45	.98	3.93	.05	.01
325	1 37.5	2 38.4	0 52.8	1.25	5.00	.07	.01
350	45	3 3.8	1 1.3	1.56	6.23	.10	.02
375	52.5	30.9	10.3	1.92	7.67	.14	.02
400	2 00	4 00	20	2.33	9.31	.19	.03
425	2 7.5	4 30.9	1 30.3	2.79	11.16	.26	.04
450	15	5 3.8	41.3	3.31	13.25	.35	.06
475	22.5	38.4	52.8	3.89	15.58	.46	.08
500	30	6 15	2 5	4.54	18.16	.59	.10
525	2 37.5	6 53.4	2 17.8	5.26	21.03	.75	.13
550	45	7 33.8	31.3	6.04	24.17	.95	.16
575	52.5	8 15.9	45.3	6.91	27.62	1.20	.20
600	3 00	9 00	3 00	7.84	31.36	1.48	.24
625	3 7.5	9 45.9	3 15.3	8.87	35.45	1.81	.30
650	15	10 33.8	31.3	9.97	39.85	2.21	.37
675	22.5	11 23.4	47.8	11.16	44.63	2.66	.44
700	30	12 15	4 4.9	12.45	49.73	3.20	.53
725	3 37.5	13 8.4	4 22.7	13.83	55.22	3.81	.64
750	45	14 3.8	41.2	15.30	61.09	4.51	.75
775	52.5	15 0.9	5 00.1	16.88	67.37	5.31	.89
800	4 00	16 0	19.8	18.56	74.05	6.22	1.04

TABLE III

TRANSITION SPIRAL

$a = \frac{4^\circ}{7}$. *1° in 175 feet*

l	d	δ	θ	F	y	x cor.	t cor.
	° ′	° ′	° ′	Feet	Feet	Feet	Feet
25	0 8.6	0 1.1	0 0.4	.00	.00	.00	.00
50	17.1	4.3	1.4	.00	.02	.00	.00
75	25.7	9.6	3.2	.02	.07	.00	.00
100	34.3	17.1	5.7	.04	17	.00	.00
125	0 43.9	0 26.7	0 8.9	.08	.32	.00	.00
150	51.4	38.6	12.9	.14	.56	.00	.00
175	1 0	52.5	17.5	.22	.89	.00	.00
200	8.6	1 8.6	22.9	.33	1.33	.01	.00
225	1 17.1	1 26.8	0 28.9	.47	1.89	.01	.00
250	25.7	47.2	35.7	.65	2.59	.02	.00
275	34.3	2 9.7	43.2	.86	3.45	.04	.01
300	42.9	34.4	51.5	1.12	4.48	.06	.01
325	1 51.4	3 1.1	1 00.4	1.42	5.70	.09	.02
350	2 0	30	10	1.78	7.12	.13	.02
375	8.6	4 1.1	20.4	2.19	8.76	.19	.03
400	17.1	34.4	31.5	2.66	10.63	.26	.04
425	2 25.7	5 9.8	1 43.3	3.18	12.74	.35	.06
450	34.3	47.3	55.8	3.78	15.11	.46	.08
475	42.9	6 26.9	2 9	4.44	17.78	.60	.10
500	51.4	7 8.7	22.9	5.19	20.73	.77	.13
525	3 0	7 52.7	2 37.5	6.00	24.00	.99	.17
550	8.6	8 38.8	52.9	6.90	27.58	1.25	.21
575	17.1	9 27	3 9	7.89	31.47	1.56	.26
600	25.7	10 17.4	25.8	8.96	35.79	1.93	.32
625	3 34.3	11 9.9	3 43.2	10.13	40.44	2.36	.39
650	42.9	12 4.6	4 1.4	11.39	45.47	2.87	.48
675	51.4	13 1.4	20.4	12.74	50.89	3.47	.57
700	4 0	14 0.4	40	14.22	56.76	4.15	.69
725	4 8.6	15 1.5	5 00.3	15.80	63.08	4.99	.83
750	17.1	16 4.7	21.4	17.48	69.81	5.91	.98
775	25.7	17 10.1	43.2	19.29	76.99	6.95	1.17
800	34.3	18 17.6	6 5.6	21.21	84.63	8.14	1.36

TABLE III

TRANSITION SPIRAL

$a = \frac{4^\circ}{7}$. *1° in 175 feet*

l	d	δ	θ	F	y	x cor.	t cor.
	° ′	° ′	° ′	Feet	Feet	Feet	Feet
25	0 8.6	0 1.1	0 0.4	.00	.00	.00	.00
50	17.1	4.3	1.4	.00	.02	.00	.00
75	25.7	9.6	3.2	.02	.07	.00	.00
100	34.3	17.1	5.7	.04	17	.00	.00
125	0 43.9	0 26.7	0 8.9	.08	.32	.00	.00
150	51.4	38.6	12.9	.14	.56	.00	.00
175	1 0	52.5	17.5	.22	.89	.00	.00
200	8.6	1 8.6	22.9	.33	1.33	.01	.00
225	1 17.1	1 26.8	0 28.9	.47	1.89	.01	.00
250	25.7	47.2	35.7	.65	2.59	.02	.00
275	34.3	2 9.7	43.2	.86	3.45	.04	.01
300	42.9	34.4	51.5	1.12	4.48	.06	.01
325	1 51.4	3 1.1	1 00.4	1.42	5.70	.09	.02
350	2 0	30	10	1.78	7.12	.13	.02
375	8.6	4 1.1	20.4	2.19	8.76	.19	.03
400	17.1	34.4	31.5	2.66	10.63	.26	.04
425	2 25.7	5 9.8	1 43.3	3.18	12.74	.35	.06
450	34.3	47.3	55.8	3.78	15.11	.46	.08
475	42.9	6 26.9	2 9	4.44	17.78	.60	.10
500	51.4	7 8.7	22.9	5.19	20.73	.77	.13
525	3 0	7 52.7	2 37.5	6.00	24.00	.99	.17
550	8.6	8 38.8	52.9	6.90	27.58	1.25	.21
575	17.1	9 27	3 9	7.89	31.47	1.56	.26
600	25.7	10 17.4	25.8	8.96	35.79	1.93	.32
625	3 34.3	11 9.9	3 43.2	10.13	40.44	2.36	.39
650	42.9	12 4.6	4 1.4	11.39	45.47	2.87	.48
675	51.4	13 1.4	20.4	12.74	50.89	3.47	.57
700	4 0	14 0.4	40	14.22	56.76	4.15	.69
725	4 8.6	15 1.5	5 00.3	15.80	63.08	4.99	.83
750	17.1	16 4.7	21.4	17.48	69.81	5.91	.98
775	25.7	17 10.1	43.2	19.29	76.99	6.95	1.17
800	34.3	18 17.6	6 5.6	21.21	84.63	8.14	1.36

TABLE IV

TRANSITION SPIRAL

$a = 0° 40'$. 1° in 150 feet

l	d	δ	θ	F	y	x cor.	t cor.
	° ′	° ′	° ′	Feet	Feet	Feet	Feet
25	0 10	0 1.3	0 0.4	.00	.00	.00	.00
50	20	5	1.7	.01	.02	.00	.00
75	30	11.3	3.8	.02	.08	.00	00
100	40	20	6.7	.05	.19	.00	.00
125	0 50	0 31.3	0 10.4	.10	.38	.00	.00
150	1 00	45	15	.16	.65	.00	.00
175	10	1 1.3	20.4	.26	1.04	.01	.00
200	20	20	26.7	.39	1.55	.01	.00
225	1 30	1 41.3	0 33.8	.55	2.21	.02	.00
250	40	2 5	41.7	.76	3.03	.03	.01
275	50	31.3	50.4	1.01	4.04	.05	.01
300	2 00	3 00	1 00	1.31	5.23	.08	.01
325	2 10	3 31.3	1 10.4	1.66	6.66	.12	.02
350	20	4 5	21.7	2.08	8.31	.18	.03
375	30	41.3	33.8	2.56	10.23	.25	.04
400	40	5 20	46.7	3.10	12.40	.35	.06
425	2 50	6 1.3	2 .4	3.72	14.88	.47	.08
450	3 00	45	15	4.41	17.66	.62	.10
475	10	7 31.3	30.4	5.19	20.76	.82	.14
500	20	8 20	46.7	6.05	24.20	1.06	.18
525	3 30	9 11.3	3 3.8	7.01	28.02	1.35	.22
550	40	10 5	21.7	8.05	32.19	1.70	.28
575	50	11 1.3	40.4	9.20	36.78	2.12	.36
600	4 00	12 0	59.9	10.45	41.76	2.63	.44
625	4 10	13 1.3	4 20.3	11.83	47.20	3.22	.54
650	20	14 5	41.6	13.29	53.05	3.93	.66
675	30	15 11.3	5 3.6	14.88	59.41	4.73	.78
700	40	16 20	26.4	16.60	66.20	5.69	.94

TABLE V

TRANSITION SPIRAL

$a = 0° 48'$. 1° in 125 feet

l	d	δ	θ	F	y	x cor.	t cor.
	° ′	° ′	° ′	Feet	Feet	Feet	Feet
25	0 12	0 1.5	0 0.5	.00	.00	.00	.00
50	24	6	2	.01	.03	.00	.00
75	36	13.5	4.5	.02	.10	.00	.00
100	48	24	8	.06	.23	.00	.00
125	1 00	0 37.5	0 12.5	.11	.46	.00	.00
150	12	54	18	.20	.79	.00	.00
175	24	1 13.5	24.5	.31	1.25	.01	.00
200	36	36	32	.47	1.86	.02	.00
225	1 48	2 1.5	0 40.5	.66	2.65	.03	.00
250	2 00	30	50	.91	3.64	.05	.01
275	12	3 1.5	1 0.5	1.21	4.84	.08	.01
300	24	36	12	1.57	6.28	.12	.02
325	2 36	4 13.5	1 24.5	2.00	7.99	.18	.03
350	48	54	38	2.49	9.97	.26	.04
375	3 00	5 37.5	52.5	3.07	12.27	.36	.06
400	12	6 24	2 8	3.72	14.88	.50	.08
425	3 24	7 13.5	2 24.5	4.47	17.85	.68	.11
450	36	8 6	42	5.31	21.18	.90	.15
475	48	9 1.5	3 0.5	6.23	24.90	1.18	.20
500	4 00	10 0	20	7.26	29.02	1.52	.25
525	4 12	11 1.5	3 40.5	8.41	33.60	1.92	.33
550	24	12 6	4 2	9.66	38.62	2.44	.41
575	36	13 13.5	4 24.5	11.02	44.08	3.07	.51
600	48	14 24	4 48	12.50	50.06	3.78	.62
625	5 00	15 37.5	5 12.5	14.15	56.55	4.63	.77
650	12	16 54	5 38	15.90	63.55	5.63	.95
675	24	18 13.5	6 4	17.80	71.09	6.81	1.13
700	36	19 36	6 32	19.84	79.20	8.13	1.36

TABLE VI

TRANSITION SPIRAL

a = 0° 54′. 1° in 111.1 feet

l	d	δ	θ	F	y	x cor.	t cor.
	° ′	° ′	° ′	Feet	Feet	Feet	Feet
20	0 10.8	0 1.1	0 0.4	.00	.00	.00	.00
40	21.6	4.3	1.4	.00	.02	.00	.00
60	32.4	9.7	3.2	.01	.06	.00	.00
80	43.2	17.3	5.8	.03	.12	.00	.00
100	54.0	27.0	9.0	.06	.26	.00	.00
120	1 4.8	0 38.9	0 13.0	.11	.45	.00	.00
140	15.6	52.9	17.6	.18	.72	.00	.00
160	26.4	1 9.1	23.0	.27	1.07	.01	.00
180	37.2	27.5	29.2	.38	1.52	.01	.00
200	48.0	48.0	36.0	.52	2.09	.02	.00
220	1 58.8	2 10.7	0 43.6	.70	2.79	.03	.00
240	2 9.6	35.5	51.8	.90	3.62	.05	.01
260	20.4	3 2.5	1 00.8	1.15	4.60	.07	.01
280	31.2	31.7	10.6	1.43	5.74	.10	.02
300	42.0	4 3.0	21.0	1.77	7.08	.15	.03
320	2 52.8	4 36.5	1 32.2	2.14	8.58	.21	.03
340	3 3.6	5 12.1	44.0	2.57	10.28	.28	.05
360	14.4	49.9	56.6	3.05	12.20	.37	.07
380	25.2	6 29.9	2 10.0	3.59	14.35	.49	.08
400	36.0	7 12.0	24.0	4.19	16.73	.63	.11
420	3 46.8	7 56.3	2 38.8	4.84	19.36	.81	.13
440	57.6	8 42.7	54.2	5.57	22.26	1.01	.17
460	4 8.4	9 31.3	3 10.4	6.36	25.42	1.27	.21
480	19.2	10 22.1	27.3	7.23	28.86	1.57	.26
500	30.0	11 15.0	3 44.9	8.16	32.61	1.92	.32
520	4 40.8	12 10.1	4 3.3	9.18	36.66	2.34	.39
540	51.6	13 7.3	22.3	10.28	41.03	2.83	.47
560	5 2.4	14 6.7	42.1	11.47	45.74	3.39	.57
580	13.2	15 8.3	5 2.6	12.73	50.76	4.03	.67
600	24.0	16 12.0	23.8	14.09	56.15	4.77	.79
620	5 34.8	17 17.9	5 45.7	15.53	61.89	5.62	.94
640	45.6	18 25.9	6 8.3	17.07	68.00	6.59	1.10
660	56.4	19 36.1	31.6	18.71	74.51	7.68	1.27
680	6 7.2	20 48.5	55.7	20.46	81.39	8.91	1.47
700	18.0	22 3.0	7 20.5	22.31	88.65	10.29	1.70

TABLE VII

TRANSITION SPIRAL

a = 1° 0′. 1° in 100 feet

l	d	δ	θ	F	y	x cor.	t cor.
	°	° ′	° ′	Feet	Feet	Feet	Feet
10	.1	0 0.3	0 0.1	.000	.000	.000	.000
20	.2	1.2	0.4	.001	.002	.000	.000
30	.3	2 7	0.9	.002	.008	.000	.000
40	.4	4.8	1.6	.005	.019	.000	.000
50	.5	7 5	2.5	.009	.036	.000	.000
60	.6	0 10 8	0 3.6	.016	.063	.000	.000
70	.7	14.7	4.9	.025	.100	.000	.000
80	.8	19.2	6.4	.037	.149	.000	.000
90	.9	24 3	8.1	.053	.212	.000	.000
100	1.0	30	10	.073	.291	.001	.000
110	1.1	0 36.3	0 12.1	.097	.387	.001	.000
120	.2	43.2	14.4	.126	.503	.002	.000
130	.3	50.7	16 9	.160	.639	.003	.000
140	.4	58.8	19.6	.199	.798	.004	.000
150	.5	1 7.5	22.5	.245	.982	.006	.001
160	1.6	1 16.8	0 25.6	.298	1.191	.008	.001
170	.7	26.7	28.9	.357	1.429	.011	.002
180	.8	37 2	32.4	.424	1.696	.014	.002
190	.9	48.3	36.1	.499	1.995	.019	.003
200	2.0	2 00	40	.582	2.327	.024	.004
210	2.1	2 12.3	0 44.1	.673	2.690	.031	.005
220	.2	25 2	48.4	.774	3.097	.039	.006
230	.3	38 7	52.9	.885	3.538	.049	.008
240	.4	52 8	57.6	1.005	4.020	.061	:010
250	.5	3 7.5	1 2.5	1.136	4.544	.074	.012
260	2.6	3 22 8	1 7.6	1.278	5.111	.090	.015
270	.7	38.7	12.9	1.431	5.724	.109	.018
280	.8	55.2	18 4	1.596	6.383	.131	.022
290	.9	4 12.3	24.1	1.773	7 091	.156	.027
300	3.0	30	30	1.963	7.850	.185	.031
310	3.1	4 48 3	1 36.1	2.166	8.66	.218	.036
320	.2	5 7.2	42.4	2.382	9.53	.255	.043
330	.3	26.7	48.9	2.612	10.45	.298	.050
340	.4	46.8	55.6	2.857	11.42	.346	.058
350	.5	6 7.5	2 2.5	3.116	12.46	.400	.067
360	3.6	6 28.8	2 9.6	3.391	13.56	.460	.077
370	.7	50.7	16.9	3.681	14.72	.528	.088
380	.8	7 13.2	24.4	3.988	15.94	.603	.100
390	.9	36.3	32.1	4.311	17.23	.686	.114
400	4.0	8 00	40	4.651	18.59	.779	.130
410	4.1	8 24.3	2 48.1	5.01	20.02	.881	.147
420	.2	49.2	56.4	5.38	21.51	.994	.166
430	.3	9 14.7	3 4.9	5.78	23.08	1.118	.186
440	.4	40.8	13.6	6.19	24.73	1.254	.209
450	.5	10 7.5	22.5	6.62	26.45	1.403	.234
460	4.6	10 34.8	3 31.6	7.07	28.24	1.57	.26
470	.7	11 2.7	40.9	7.54	30.12	1.74	.29
480	.8	31.2	50.4	8.03	32.07	1.94	.32
490	.9	12 00.3	4 00.1	8.54	34.11	2.15	.36
500	5.0	30	10	9.07	36 23	2.37	.40
510	5.1	13 00.3	4 20.1	9.63	38.44	2.62	.44
520	.2	31.2	30.4	10.20	40.73	2.89	.48
530	.3	14 2.7	40.9	10.80	43.12	3.17	.53
540	.4	34.8	51.4	11.42	45.59	3.49	.58
550	.5	15 7.5	5 2.3	12.07	48.15	3.82	.64
560	5.6	15 40.8	5 13.4	12.74	50.83	4.18	.70
570	.7	16 14.7	24.7	13.43	53.56	4.56	.76
580	.8	49.2	36.2	14.14	56.40	4.98	.83
590	.9	17 24.3	47.8	14.89	59.34	5.42	.90
600	6.0	18 00	59.7	15.65	62.39	5.89	.98

ILT 281—A

TABLE VIII

TRANSITION SPIRAL

a = 1° 6′. 1° in 90.9 feet

l	*d*	δ	θ	*F*	*y*	*x* cor.	*t* cor.
	° ′	° ′	° ′	Feet	Feet	Feet	Feet
10	0 6.6	0 .3	0 0.1	.00	.00	.00	.00
20	13.2	1.3	0.4	.00	.00	.00	.00
30	19 8	3.0	1.0	.00	.01	.00	.00
40	26.4	5.3	1 8	.01	.02	.00	.00
50	33.0	8.2	2.7	.01	.04	.00	.00
60	0 39.6	0 11.9	0 4.0	.02	.07	.00	.00
70	46.2	16.1	5 4	.03	.11	.00	.00
80	52.8	21 1	7 0	.04	.16	.00	.00
90	59.4	26 7	8.9	.06	.23	.00	.00
100	1 6.0	33.0	11.0	.08	.32	.00	.00
110	1 12.6	0 39.9	0 13.3	.11	.43	.00	.00
120	19.2	47.5	15 8	.14	.55	.00	.00
130	25.8	55.8	18.6	.18	.70	.00	.00
140	32.4	1 4 7	21 6	.22	.88	.00	.00
150	39.0	14.2	24 7	.27	1.08	.01	.00
160	1 45.6	1 24.5	0 28.2	.33	1.31	.01	.00
170	52.2	35.4	31.8	.39	1.57	.01	.00
180	58 8	46.9	35.6	.47	1.87	.02	.00
190	2 5 4	59.1	39.7	.55	2.19	.02	.00
200	12.0	2 12.0	44.0	.64	2.56	.03	.00
210	2 18.6	2 25.5	0 48.5	.74	2.96	.04	.01
220	25.2	39.7	53.2	.85	3.41	.05	.01
230	31.8	54.6	58.2	.97	3.89	.06	.01
240	38.4	3 10.1	1 3.4	1.11	4.42	.07	.01
250	45.0	26.2	8.7	1.25	5.00	.09	.01
260	2 51.6	3 43.1	1 14.3	1.40	5.62	.11	.02
270	58.2	4 00.6	20.2	1.57	6.30	.13	.02
280	3 4.8	18.7	26.2	1.76	7.02	.16	.03
290	11.4	37.5	32.5	1.95	7.80	.19	.03
300	18.0	57 0	39.0	2.16	8.63	.22	.04
310	3 24.6	5 17.1	1 45.7	2.38	9.53	.26	.04
320	31.2	37.9	52.6	2.62	10.48	.31	.05
330	37.8	59.4	59.8	2.87	11.49	.36	.06
340	44.4	6 21.5	2 7.2	3.14	12.56	.42	.07
350	51.0	44.2	14.7	3.43	13.71	.49	.08
360	3 57.6	7 7.7	2 22.6	3.73	14.92	.56	.09
370	4 4.2	7 31.8	30.6	4.05	16.19	.64	.11
380	10.8	56.5	38.8	4.39	17.53	.73	.12
390	17.4	8 21.9	47.3	4.74	18.95	.83	.14
400	24.0	48.0	56.0	5.12	20.45	.94	.16
410	4 30.6	9 14.7	3 4.9	5.51	22.02	1.07	.18
420	37.2	42.1	14.0	5.92	23.66	1.20	.20
430	43.8	10 10.2	23.4	6.36	25.39	1.36	.23
440	50.4	38.9	32.9	6.81	27.20	1.52	.25
450	57.0	11 8.2	42.6	7.28	29.09	1.70	.28
460	5 3.6	11 38.3	3 52.7	7 78	31.06	1.90	.31
470	10.2	12 9.0	4 2.9	8.29	33.13	2.11	.35
480	16.8	40.3	13.3	8.83	35.28	2.35	.39
490	23.4	13 12.3	24.0	9.39	37.52	2.60	.43
500	30.0	45.0	34.9	9.98	39.86	2.87	.48
510	5 36.6	14 18.3	4 26.0	10.59	42.28	3.17	.53
520	43.2	52.3	37.3	11.22	44.80	3.50	.58
530	49.8	15 27.0	5 8.8	11.88	47.43	3.84	.64
540	56.4	16 2.3	20.6	12.56	50.15	4.22	.70
550	6 3.0	38.2	32.5	13.28	52.97	4.62	.77
560	6 9.6	17 14.9	5 44.8	14.01	55.91	5.06	.85
570	16.2	51.2	57.1	14.77	58.92	5.52	.92
580	22.8	18 30.1	6 9.7	15.55	62.04	6.03	1.00
590	29.4	19 8.7	22.6	16.38	65.27	6.56	1.09
600	36.0	48 0	35.6	17.21	68.63	7.13	1.19

TABLE IX

TRANSITION SPIRAL

a = 1° 15′. 1° in 80 *feet*

l	*d*	δ	θ	*F*	*y*	*x* cor.	*t* cor.
	° ′	° ′	° ′	Feet	Feet	Feet	Feet
10	0 7.5	0 0.5	0 0	.00	.00	.0	.0
20	15	1.5	0.5	.00	.00	.0	.0
30	22.5	3.5	1	.00	.01	.0	.0
40	30	6	2	.00	.02	.0	.0
50	37.5	9.5	3	.01	.04	.0	.0
60	0 45	0 13.5	0 4.5	.02	.08	.0	.0
70	52.5	18.5	6	.03	.12	.0	.0
80	1 0	24	8	.05	.19	.0	.0
90	7.5	30.5	10	.07	.26	.0	.0
100	15	37.5	12.5	.09	.36	.0	.0
110	1 22.5	0 45.5	0 15	.12	.48	.0	.0
120	30	54	18	.16	.63	.0	.0
130	37.5	1 3.5	21	.20	.80	.0	.0
140	45	13.5	24.5	.25	1.00	.0	.0
150	52.5	24.5	28	.31	1.23	.0	.0
160	2 0	1 36	0 32	.37	1.49	.0	.0
170	7.5	48.5	36	.45	1.77	.0	.0
180	15	2 1.5	40.5	.53	2.12	.0	.0
190	22.5	15.5	45	.62	2.50	.0	.0
200	30	30	50	.73	2.90	.0	.0
210	2 37.5	2 45.5	0 55	.84	3.36	.0	.0
220	45	3 1.5	1 00.5	.97	3.87	.0	.0
230	52.5	18.5	6	1.10	4.42	.0	.0
240	3 0	36	12	1.25	5.02	.0	.0
250	7.5	54.5	18	1.42	5.67	.1	.0
260	3 15	4 13.5	1 24.5	1.59	6.38	.1	.0
270	22.5	33.5	31	1.79	7.15	.2	.0
280	30	54	38	1.99	7.98	.2	.0
290	37.5	5 15.5	45	2.21	8.86	.2	.0
300	45	37.5	52.5	2.45	9.81	.3	.0
310	3 52.5	6 .5	2 0	2.70	10.74	.3	.0
320	4 0	24	8	2.98	11.91	.4	.0
330	7.5	48.5	16	3.26	13.06	.4	.0
340	15	7 13.5	24.5	3.57	14.28	.5	.0
350	22.5	39.5	33	3.89	15.57	.6	.0
360	4 30	8 6	2 42	4.23	16.95	.7	.1
370	37.5	33.5	51	4.59	18.40	.8	.1
380	45	9 1.5	3 00.5	4.97	19.92	.9	.2
390	52.5	30.5	10	5.38	21.54	1.0	.2
400	5 00	10 00	20	5.80	23.23	1.2	.2
410	5 7.5	10 30.5	3 30	6.26	25.00	1.4	.2
420	15	11 1.5	40.5	6.72	26.86	1.6	.3
430	22.5	33.5	51	7.22	28.82	1.7	.3
440	30	12 06	4 02	7.74	30.87	2.0	.3
450	37.5	39.5	13	8.28	33.02	2.2	.4
460	5 45	13 13.5	4 24.5	8.84	35.25	2.4	.4
470	52.5	48.5	36	9.41	37.59	2.7	.5
480	6 00	14 24	48	10.03	40.02	3.0	.5
490	7.5	15 00.5	5 00	10.67	42.56	3.4	.6
500	15	37.5	12.5	11.33	45.20	3.7	.6
510	6 22.5	16 15.5	5 25	12.03	47.95	4.1	.7
520	30	54	38	12.74	50.79	4.5	.8
530	37.5	17 33.5	51	13.48	53.76	5.0	.8
540	45	18 13.5	6 4	14.26	56.84	5.4	.9
550	52.5	54.5	18	15.07	60.02	6.0	1.0
560	7 0	19 36	6 32	15.90	63.34	6.5	1.1
570	7.5	20 18.5	46	16.76	66.72	7.1	1.2
580	15	21 1.5	7 00	17.65	70.26	7.8	1.3
590	22.5	45.5	14.5	18.57	73.90	8.4	1.4
600	30	22 30	29	19.52	77.68	9.2	1.5

TABLE X

TRANSITION SPIRAL

a = 1° 40′. 1° in 60 feet

l	d	δ	θ	F	y	x cor.	t cor.
	° ′	° ′	° ′	Feet	Feet	Feet	Feet
10	0 10	0 0.5	0 0	.00	.00	.0	.0
20	20	2	0.5	.00	.00	.0	.0
30	30	4.5	1.5	.00	.00	.0	.0
40	40	8	3	.00	.03	.0	.0
50	50	12.5	4	.00	.06	.0	.0
60	1 00	0 18	0 6	.03	.10	.0	.0
70	10	24.5	8	.04	.17	.0	.0
80	20	32	10.5	.06	.25	.0	.0
90	30	40.5	13 5	.09	.35	.0	.0
100	40	50	16.5	.12	.48	.0	.0
110	1 50	1 00.5	0 20	.16	.64	.0	.0
120	2 00	12	24	.21	.84	.0	.0
130	10	24.5	28	.26	1.06	.0	.0
140	20	38	32.5	.33	1.33	.0	.0
150	30	52.5	37 5	.41	1.63	.0	.0
160	2 40	2 8	0 42.5	.50	1.98	.0	.0
170	50	24.5	48	.59	2.38	.0	.0
180	3 00	42	54	.70	2.82	.0	.0
190	10	3 00.5	1 00	.83	3.32	.0	.0
200	20	20	6.5	.97	3.88	.0	.0
210	3 30	3 40.5	1 13.5	1.12	4.48	.1	.0
220	40	4 2	20.5	1.29	5.15	.1	.0
230	50	24.5	28	1.47	5.90	.1	.0
240	4 00	48	36	1.67	6.69	.2	.0
250	10	5 12.5	44	1.89	7.58	.2	.0
260	4 20	5 38	1 52.5	2.13	8.52	.2	.0
270	30	6 4.5	2 1.5	2.38	9.54	.3	.0
280	40	32	10.5	2.65	10.64	.4	.0
290	50	7 00.5	20	2.94	11.82	.4	.0
300	5 00	30	30	3.26	13.07	.5	.0
310	5 10	8 00.5	2 40	3.60	14.43	.6	.1
320	20	32	50.5	3.96	15.87	.7	.1
330	30	9 04.5	3 1.5	4.34	17.40	.8	.1
340	40	38	12.5	4.75	19.02	.9	.2
350	50	10 12.5	24	5.18	20.74	1.1	.2
360	6 00	10 48	3 36	5.64	22.56	1.3	.2
370	10	11 24.5	48	6.12	24.50	1.4	.2
380	20	12 2	4 00.5	6.63	26.53	1.7	.3
390	30	40.5	13.5	7.16	28.67	1.9	.3
400	40	13 20	26.5	7.73	30.92	2.2	.4
410	6 50	14 00.5	4 40	8.34	33.27	2.4	.4
420	7 00	42	54	8.96	35.73	2.8	.5
430	10	15 24.5	5 8	9.61	38.32	3.1	.5
440	20	16 8	22.5	10.30	41.07	3.5	.6
450	30	52.5	37.5	11.01	43.90	3.9	.6
460	7 40	17 38	5 52	11.75	46.86	4.3	.7
470	50	18 24.5	6 8	12.50	49.94	4.8	.8
480	8 00	19 12	24	13.35	53.16	5.4	.9
490	10	20 00.5	40	14.19	56.52	5.9	1.0
500	20	50	56	15.07	60.01	6.6	1.1
510	8 30	21 40.5	7 13	16.00	63.64	7.2	1.2
520	40	22 32	30	16.94	67.36	8.0	1.3
530	50	23 24.5	47.5	17.93	71.25	8.8	1.5
540	9 00	24 18	8 5	18.95	75.31	9.6	1.6
550	10	25 12.5	23	20.03	79.53	10.5	1.8
560	9 20	26 8	8 42	21.13	83.88	11.5	1.9
570	30	27 4.5	9 00.5	22.26	88.31	12.6	2.1
580	40	28 2	19.5	23.42	92.92	13.7	2.3
590	50	29 00.5	39	24.67	97.70	14.9	2.5
600	10 00	30 00	59	25.91	102.66	16.2	2.7

TABLE XI

TRANSITION SPIRAL

$a = 2° 0'$. 1° in 50 feet

l	d	δ	θ	F	ν	x cor.	t cor.
	° ′	° ′	° ′	Feet	Feet	Feet	Feet
10	0 12	0 0.5	0 0	.00	.00	.0	.0
20	24	2.5	1	.00	.00	.0	.0
30	36	5.5	2	.00	.02	.0	.0
40	48	9.5	3	.01	.04	.0	.0
50	1 00	15	5	.02	.07	.0	.0
60	1 12	0 21.5	0 7	.03	.13	.0	.0
70	24	29.5	10	.05	.20	.0	.0
80	36	38.5	13	.07	.30	.0	.0
90	48	48.5	16	.10	.42	.0	.0
100	2 00	1 00	20	.15	.58	.0	.0
110	[illegible] 12	1 12.5	0 24	.19	.77	.0	.0
120	24	26.5	29	.25	1.00	.0	.0
130	36	41.5	34	.32	1.28	.0	.0
140	48	57.5	39	.40	1.60	.0	.0
150	3 00	2 15	45	.49	1.96	.0	.0
160	3 12	2 33.5	0 51	.59	2.38	.0	.0
170	24	53.5	58	.71	2.86	.0	.0
180	36	3 14.5	1 5	.85	3.39	.1	.0
190	48	36.5	12	1.00	3.99	.1	.0
200	4 00	4 00	20	1.16	4.65	.1	.0
210	4 12	4 24.5	1 28	1.35	5.38	.1	.0
220	24	50.5	37	1.54	6.19	.2	.0
230	36	5 17.5	46	1.76	7.07	.2	.0
240	48	45.5	55	2.00	8.04	.2	.0
250	5 00	6 15	2 5	2.27	9.09	.3	.0
260	5 12	6 45.5	2 15	2.55	10.22	.4	.0
270	24	7 17.5	26	2.85	11.45	.4	.0
280	36	50.5	37	3.18	12.75	.5	.0
290	48	8 24.5	48	3.54	14.18	.6	.1
300	6 00	9 00	3 00	3.91	15.68	.7	.1
310	6 12	9 36.5	3 12	4.32	17.31	.9	.1
320	24	10 14.5	25	4.75	19.03	1.0	.2
330	36	53.5	38	5.21	20.87	1.2	.2
340	48	11 33.5	51	5.70	22.81	1.4	.2
350	7 00	12 15	4 5	6.22	24.87	1.6	.3
360	7 12	12 57.5	4 19	6.77	27.05	1.8	.3
370	24	13 41.5	34	7.34	29.35	2.1	.3
380	36	14 26.5	49	7.95	31.79	2.4	.4
390	48	15 12.5	5 4	8.60	34.35	2.7	.4
400	8 00	16 00	20	9.28	37.04	3.1	.5
410	8 12	16 48.5	5 36	10.00	39.85	3.5	.6
420	24	17 38.5	53	10.73	42.79	4.0	.7
430	36	18 29.5	6 10	11.53	45.88	4.4	.7
440	48	19 21.5	27	12.34	49.14	5.0	.8
450	9 00	20 15	45	13.20	52.55	5.6	.9
460	9 12	21 9.5	7 3	14.09	56.05	6.3	1.0
470	24	22 5.5	21	15.02	59.73	6.9	1.2
480	36	23 2.5	40	15.99	63.55	7.7	1.3
490	48	24 0.5	8 00	17.00	67.55	8.5	1.4
500	10 00	25 00	19	18.05	71.72	9.4	1.6
510	10 12	26 0.5	8 39	19.15	76.00	10.4	1.7
520	24	27 2.5	9 00	20.27	80.04	11.4	1.9
530	36	28 5.5	21	21.45	85.08	12.6	2.1
540	48	29 9.5	42	22.68	89.88	13.8	2.3
550	11 00	30 15	10 3.5	23.96	94.85	15.1	2.5
560	11 12	31 21.5	10 26	25.27	99.97	16.5	2.8
570	24	32 29.5	48	26.62	105.19	18.0	3.0
580	36	33 38.5	11 10.5	28.01	110.62	19.6	3.3
590	48	34 48.5	34	29.48	116.27	21.3	3.6
600	12 00	36 00	58	30.97	122.13	23.2	3.9

TABLE XII

TRANSITION SPIRAL

a = 2° 30′. 1° in 40 feet

l	d	δ	θ	F	y	x cor.	t cor.
	° ′	° ′	° ′	Feet	Feet	Feet	Feet
10	0 15	0 1	0 0	.00	.00	.0	.0
20	30	3	1	.00	.00	.0	.0
30	45	7	2	.00	.02	.0	.0
40	1 00	12	4	.01	.05	.0	.0
50	15	19	6	.02	.09	.0	.0
60	1 30	0 27	0 9	.04	.16	.0	.0
70	45	37	12	.06	.25	.0	.0
80	2 00	48	16	.09	.37	.0	.0
90	15	1 1	20	.13	.53	.0	.0
100	30	15	25	.18	.73	.0	.0
110	2 45	1 31	0 30	.24	.97	.0	.0
120	3 00	48	36	.31	1.25	.0	.0
130	15	2 7	42	.40	1.60	.0	.0
140	30	27	49	.50	2.00	.0	.0
150	45	49	56	.61	2.45	.0	.0
160	4 00	3 12	1 4	.74	2.97	.0	.0
170	15	37	12	.89	3.57	.0	.0
180	30	4 3	21	1.06	4.24	.1	.0
190	45	31	30	1.25	4.99	.1	.0
200	5 00	5 00	40	1.45	5.81	.2	.0
210	5 15	5 31	1 50	1.68	6.72	.2	.0
220	30	6 3	2 1	1.93	7.74	.2	.0
230	45	37	12	2.20	8.85	.3	.0
240	6 00	7 12	24	2.51	10.05	.4	.0
250	15	49	36	2.84	11.37	.5	.1
260	6 30	8 27	2 49	3.19	12.77	.6	.1
270	45	9 7	3 2	3.57	14.29	.7	.1
280	7 00	48	16	3.98	15.94	.8	.1
290	15	10 31	30	4.42	17.70	1.0	.2
300	30	11 15	45	4.89	19.59	1.2	.2
310	7 45	12 1	4 00	5.40	21.61	1.4	.2
320	8 00	48	16	5.94	23.76	1.6	.3
330	15	13 37	32	6.51	26.05	1.9	.3
340	30	14 27	49	7.12	28.46	2.2	.4
350	45	15 19	5 6	7.77	31.03	2.5	.4
360	9 00	16 12	5 24	8.46	33.74	2.9	.5
370	15	17 7	42	9.18	36.62	3.3	.5
380	30	18 3	6 1	9.95	39.64	3.7	.6
390	45	19 1	20	10.75	42.82	4.3	.7
400	10 00	20 00	40	11.60	46.16	4.9	.8
410	10 15	21 1	7 00	12.47	40.65	5.5	.9
420	30	22 3	21	13.39	53.28	6.2	1.0
430	45	23 7	42	14.38	57.10	6.9	1.2
440	11 00	24 12	8 4	15.39	61.12	7.8	1.3
450	15	25 19	26	16.45	65.32	8.7	1.5

TABLE XIII

TRANSITION SPIRAL

$a = 3° 20'$. 1° in 30 feet

l	d	δ	θ	F	y	x cor.	t cor.
	° ′	° ′	° ′	Feet	Feet	Feet	Feet
10	0 20	0 1	0 0	.00	.00	.0	.0
20	40	4	1	.00	.01	.0	.0
30	1 00	9	3	.01	.03	.0	.0
40	20	16	5	.02	.06	.0	.0
50	40	25	8	.03	.12	.0	.0
60	2 00	0 36	0 12	.05	.21	.0	.0
70	20	49	16	.08	.33	.0	.0
80	40	1 4	21	.12	.50	.0	.0
90	3 00	21	27	.18	.71	.0	.0
100	20	40	33	.24	.97	.0	.0
110	3 40	2 1	0 40	.32	1.29	.0	.0
120	4 00	24	48	.42	1.68	.0	.0
130	20	49	56	.53	2.13	.0	.0
140	40	3 16	1 5	.67	2.66	.0	.0
150	5 00	45	15	.82	3.27	.1	.0
160	5 20	4 16	1 25	.99	3.97	.1	.0
170	40	49	36	1.19	4.76	.1	.0
180	6 00	5 24	48	1.41	5.65	.2	.0
190	20	6 1	2 00	1.66	6.65	.2	.0
200	40	40	13	1.94	7.75	.3	.0
210	7 00	7 21	2 27	2.24	8.97	.3	.1
220	20	8 4	41	2.58	10.31	.4	.1
230	40	49	56	2.95	11.77	.5	.1
240	8 00	9 36	3 12	3.35	13.38	.7	.1
250	20	10 25	28	3.78	15.11	.8	.1
260	8 40	11 16	3 45	4.25	17.00	1.0	.2
270	9 00	12 9	4 3	4.76	19.02	1.2	.2
280	20	13 4	21	5.31	21.20	1.4	.2
290	40	14 1	40	5.90	23.55	1.7	.3
300	10 00	15 00	5 00	6.53	26.05	2.0	.3
310	10 20	16 1	5 20	7.20	28.72	2.4	.4
320	40	17 4	41	7.92	31.57	2.8	.5
330	11 00	18 9	6 3	8.69	34.59	3.3	.5
340	20	19 16	25	9.49	37.80	3.8	.6
350	40	20 25	48	10.35	41.19	4.4	.7
360	12 00	21 36	7 11	11.25	44.78	5.1	.8
370	20	22 49	36	12.21	48.56	5.8	1.0
380	40	24 4	8 00	13.22	52.53	6.6	1.1
390	13 00	25 21	26	14.28	56.71	7.6	1.3
400	20	26 40	52	15.39	61.10	8.6	1.4
410	13 40	28 1	9 19	16.56	65.69	9.7	1.6
420	14 00	29 24	47	17.79	70.49	10.9	1.8
430	20	30 49	10 15	19.07	75.51	12.3	2.1
440	40	32 16	43	20.41	80.74	13.7	2.3
450	15 00	33 45	11 13	21.81	86.19	15.4	2.6

TABLE XIV

TRANSITION SPIRAL

a = 5° 0′. 1° in 20 feet

l	d	δ	θ	F	y	x cor.	t cor.
	° ′	° ′	° ′	Feet	Feet	Feet	Feet
10	0 30	0 1	0 0	.00	.00	.0	.0
20	1 00	6	2	.00	.01	.0	.0
30	30	13	4	.01	.04	.0	.0
40	2 00	24	8	.02	.09	.0	.0
50	30	37	12	.05	.18	.0	.0
60	3 00	0 54	0 18	.08	.31	.0	.0
70	30	1 13	24	.12	.50	.0	.0
80	4 00	36	32	.19	.74	.0	.0
90	30	2 1	40	.26	1.06	.0	.0
100	5 00	30	50	.36	1.45	.0	0
110	5 30	3 1	1 00	.48	1.94	.0	0
120	6 00	36	12	.62	2.51	.0	.0
130	30	4 13	24	.79	3.20	.0	.0
140	7 00	54	38	.99	3 99	.1	.0
150	30	5 37	52	1.22	4.90	.1	.0
160	8 00	6 24	2 8	1.48	5.96	.2	.0
170	30	7 13	24	1.78	7.15	.3	.0
180	9 00	8 6	42	2.11	8.49	.4	.0
190	30	9 1	3 00	2.49	9.98	.5	.0
200	10 00	10 0	20	2.90	11.62	.6	.1
210	10 30	11 1	3 40	3.36	13.45	.8	.1
220	11 00	12 6	4 2	3.86	15.44	1.0	.2
230	30	13 13	24	4.41	17.63	1.2	.2
240	12 00	14 24	48	5.01	20.01	1.5	.3
250	30	15 37	5 12	5.66	22.60	1.8	.3
260	13 00	16 54	5 38	6.37	25.38	2.2	.4
270	30	18 13	6 4	7.12	28.39	2.7	.5
280	14 00	19 36	32	7.94	31.62	3.3	.6
290	30	21 2	7 00	8.82	35.10	3 9	.7
300	15 00	22 30	29	9.76	38 83	4.6	.8
310	15 30	24 2	8 00	10.76	42.73	5.4	.9
320	16 00	25 36	31	11.82	46.92	6.3	1.1
330	30	27 13	9 04	12.95	51.36	7.4	1.2
340	17 00	28 54	37	14.15	56.05	8.6	1.4
350	30	30 37	10 11	15.43	61.09	9.9	1.7
360	18 00	32 24	10 46	16.75	66.31	11.3	1.9
370	30	34 14	11.19	18.16	71.63	13.0	2.2
380	19 00	36 6	12 00	19 65	77.35	14 8	2 5
390	30	38 2	38	21.21	83.41	16.8	2.8
400	20 00	40 0	13 17	22.87	89.83	19 0	3.2

TABLE XV
TRANSITION SPIRAL
a = 10° 0'. 1° in 10 feet

l	*d*	*δ*	*θ*	*F*	*y*	*x* cor.	*t* cor.
	° ′	° ′	° ′	Feet	Feet	Feet	Feet
10	1 00	0 3	0 1	.00	.00	.0	.0
20	2	12	4	.01	.02	.0	.0
30	3	27	9	.02	.08	.0	.0
40	4	48	16	.05	.19	.0	.0
50	5	1 15	25	.09	.36	.0	.0
60	6 00	1 48	0 36	.16	.63	.0	.0
70	7	2 27	49	.25	1.00	.0	.0
80	8	3 12	1 4	.37	1.49	.0	.0
90	9	4 3	21	.53	2.12	.0	.0
100	10	5 0	40	.73	2.91	.1	.0
110	11 00	6 3	2 1	.97	3.87	.1	.0
120	12	7 12	24	1.26	5.02	.2	.0
130	13	8 27	49	1.60	6.38	.3	.0
140	14	9 48	3 16	1.99	7.97	.4	.1
150	15	11 15	45	2.45	9.79	.6	.1
160	16 00	12 48	4 16	2.97	11.87	.8	.1
170	17	14 27	49	3.56	14.23	1.1	.2
180	18	16 12	5 24	4.23	16.87	1.4	.2
190	19	18 3	6 1	4.97	19.81	1.9	.3
200	20	20 0	39	5.79	23.07	2.4	.4
210	21 00	22 3	7 20	6.70	26.65	3.1	.5
220	22	24 12	8 3	7.69	30.58	3.9	.6
230	23	26 27	48	8.78	34.86	4.8	.8
240	24	28 48	9 35	9.96	39.49	6.0	1.0
250	25	31 15	10 23	11.24	44.49	7.3	1.2
260	26 00	33 48	11 14	12.61	49.67	8.9	1.5
270	27	36 27	12 7	14.07	55.30	10.8	1.8
280	28	39 12	13 1	15.67	61.40	12.9	2.1
290	29	42 3	57	17.39	67.97	15.3	2.6
300	30	45 0	14 55	19.23	75.07	18.1	3.1

EARTHWORK

EARTHWORK SURVEYS

CUTS AND FILLS

1. Necessity for Cuts and Fills.—Economical operation of a railroad, as well as of a highway, requires that the irregularities of the natural surface of the ground be equalized as far as possible by means of cuts and fills. These changes of grade involve extensive earthwork, particularly in the construction of a railroad, which should be level or as nearly so as possible. The following discussion is devoted mainly to earthwork in railroad construction, but the principles explained apply equally to highway work.

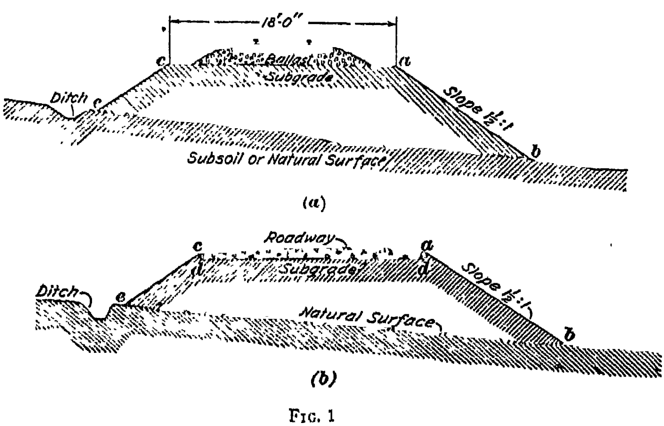

FIG. 1

2. A **fill,** Fig. 1, is an embankment that supports the roadbed above the natural surface. A fill in railroad work is shown in (*a*) and a fill in highway work in (*b*).

3. A **cut,** Fig. 2, is an excavation that permits the roadbed to be placed below the natural surface. This definition is not intended to include tunnels. A cross-section of a railway in cut is shown in (*a*) and of a highway in (*b*).

4. The **subgrade,** Figs. 1 and 2, is the natural surface on which the road metal in highway work, or the ballast in railroad work, rests.

5. The **subsoil** is either the natural soil on which an embankment rests, as in Fig. 1, or the natural soil under the subgrade of an excavation, as in Fig. 2.

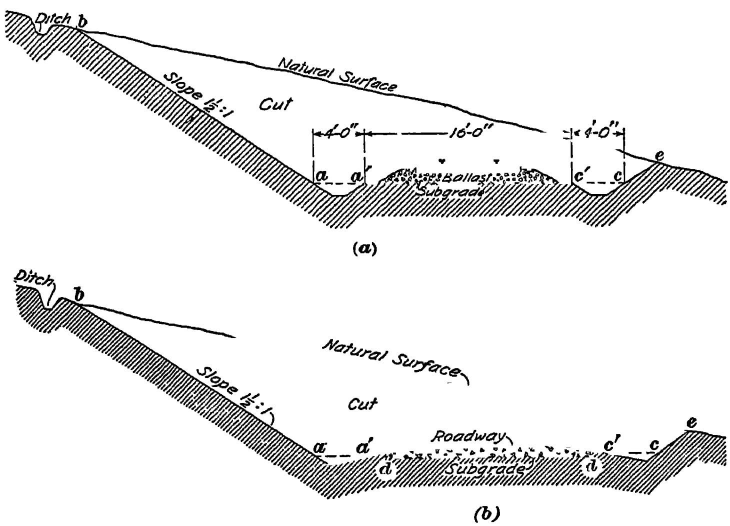

FIG. 2

6. The term **roadbed** in railroad work is usually applied to the subgrade and ballast; it includes the ditches in a cut. The phrase *width of roadbed* means the width *a c*, Figs. 1 (*a*) and 2 (*a*). In highway work the term roadbed is applied to the natural foundation of the roadway, and the phrase *width of roadbed* means the width *d d*, Figs. 1 (*b*) and 2 (*b*).

7. **Grade** is a term applied to the longitudinal slope or inclination of the roadway or track. The term is also used, in contradistinction to the term subgrade, to denote the base

of the rails or the upper surface of a highway. The expression *level grade* is often used in the sense of level track, or level road.

8. By the **side slope,** or simply the **slope,** of a cut or fill is meant the inclination of the sides (*a b* and *c e,* Figs. 1 and 2) of the cut or fill to the vertical. A side slope is usually indicated by stating the rate at which the side of the cut or fill diverges from the vertical. This rate is called the *rate of slope,* or *slope ratio.* Thus, a slope of 2 to 1 is one in which the side diverges from the vertical at the rate of 2 units of length measured horizontally in every unit of length measured vertically. For example, in Fig. 3, which shows different

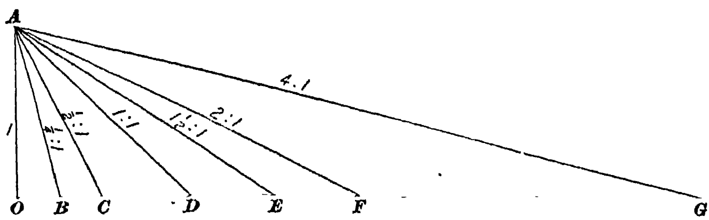

FIG. 3

rates of slope, the ratio 2 : 1 marked on *A F* indicates that, in the vertical distance *A O* = 1, the horizontal distance *O F* by which the line *A F* diverges from the vertical, is 2. It will be observed that the slope, or rate of slope, of a line is the tangent of the angle that the line makes with the vertical. Thus, in the case of *A C*, Fig. 3, the slope ½ : 1, is equal to $O\,C \div O\,A = \tan O\,A\,C$. In the following the rate of slope will be denoted by *s*.

9. Slope Ratio in Cuts.—The very hardest and firmest rock may sometimes be safely cut out so as to leave a vertical face of wall. It should be observed, however, that rock which appears very hard and firm when first excavated often contains seams that will be opened by the action of frost which may cause large pieces to break
care should be taken to di
This will practically mean
average slope of about ¼ : 1

the slope must be flattened, until for a soil of firm earth or gravel a slope of 1:1 may be permissible for cuts, although a slope of $1\frac{1}{2}$: 1 is commonly adopted, especially if the soil is soft and liable to wash. As stated before, a very soft and treacherous soil may require that the slope ratio be cut down even as flat as 4 : 1.

10. Slope Ratio in Fills.—A fill is usually made from the material excavated in an adjoining cut. But if it should happen that the quality of the soil is such that it is liable to slide, it may prove to be an economy to reject such soil by *wasting* it, even though it may be necessary to *borrow* a better grade of soil from some place in the neighborhood of the fill. An earthwork fill is generally made with a slope ratio of $1\frac{1}{2}$: 1. This may be considered standard practice. When a fill is made from the material taken from a rock cut, it may be possible to make a stable embankment with a slope ratio of 1 : 1. On side-hill work, where a slope ratio of $1\frac{1}{2}$: 1, or even 1 : 1, might require a very long slope, it may often be advisable to make a rough dry wall of the stones from a rock cut, which will have a slope ratio of $\frac{3}{4}$: 1, or steeper.

11. Width of Excavations and Embankments.—The width required for a standard-gauge single-track roadbed on a fill, may be estimated as follows: The tie will be about 8 feet 6 inches long. At the ends of the ties, the ballast will slope down to subgrade. The extra width required for this varies with the kind of ballast used, but it will be about 1 or 2 feet at each end of the tie. Usually, the embankment is widened for about 2 feet beyond the ballast on each side. The absolute minimum for the width of subrade for a fill is, therefore, $8\frac{1}{2}$ feet $+ 2 \times (1 + 2)$ feet, or about $14\frac{1}{2}$ feet. This width would only be used for light-traffic, cheaply-constructed roads; 16 to 18 feet is far more common, while 20 feet and even more is frequently used, as the danger of accident due to a washing out of the embankment is materially reduced by widening the roadbed. In cuts these figures should be increased from 6 to 8 feet to include the two side ditches. When excavation is made through rock, the danger of scouring

during heavy rain storms being eliminated, the total required width may be very materially reduced from the figures just given. The heavy expense of excavating through solid rock requires that such economy shall be used if possible.

12. Ditching.—The great enemy of track maintenance is water, because it not only scours away the subsoil and ballast, but also freezes in winter, heaves the soil, and produces a rough track, which becomes a soft track when it thaws out. It is, therefore, of the utmost importance that adequate ditches should be provided to carry away quickly from the roadbed all rain water that may fall on or near it. Ditches should be constructed on both sides of the track through cuts. The bottom of the ditch should be enough lower than the subgrade to drain the water from it. A ditch should also be constructed at the top of a cut, so as to catch all the water that may come down the natural slope above the cut and prevent it from washing down the side of the cut; this ditch is shown at *b*, Fig. 2. All such ditches should lead off to some water course, if possible, or at least to some point where the outfall may not cause any scour. If the soil is very soft and the amount of water that will go through any one ditch is very large, it may cause such a scour that paving the ditch becomes economical.

FIELD WORK

13. The first step in the work of construction is to clear off all growth of timber within the limits of the right of way. The engineer with his party passes over the line, making offsets to the right and to the left, and blazing the trees that stand on, or just within, the limits of the company's property. The blazed spot is marked with a letter C, as a guide to the contractor. The valuable timber, when felled, should be piled near the boundary lines, to be saved as the property of the company; the brushwood should be burned. Where a deep cut is to be made, the stumps are left to be removed as the earth is excavated. In very shallow cuts and fills, the

contractor will generally prefer to tear up the trees by the roots at once, rather than to grub out the stumps after clearing. Where the embankments will be over 3 feet high, grubbing is not necessary; but the trees require to be *low-chopped*, leaving no stumps above the roots. The engineer should indicate to the contractor the localities where each process is suitable.

While the clearing is in progress, the engineer should run a line of test levels touching on all the benches to verify their elevations; he may also rerun the center line, replacing any stakes that have disappeared, and setting additional stakes wherever the inclination of the natural surface along the center line changes abruptly. If any changes in the alinement have been ordered, these should be made at the same time.

14. The Grade Profile.—The engineer is furnished with a profile of the line on which the established grade is indicated. This established grade on the profile consists of a series of straight lines, the elevations of the ends of which should be clearly indicated. These elevations are the elevations of the subgrade, Figs. 1 and 2.

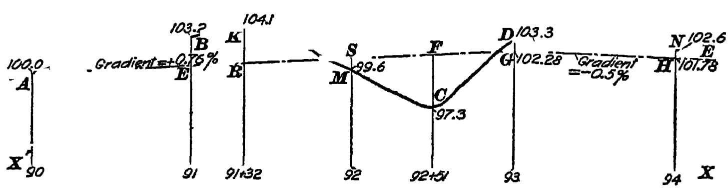

FIG. 4

A short portion of a profile is shown in Fig. 4. The horizontal line XX' represents any horizontal plane, and the broken line AGH shows the position of the established grade. The station numbers are written along the line XX', and the elevations of the corresponding points of the established grade are written along the grade line. Thus, in Fig. 4, the elevation of subgrade at Sta. 90, or A, is 100 feet; at Sta. 93, or G, it is 102.28 feet; and at Sta. 94, or H, it is 101.78 feet.

The **gradient** of the established grade is the number of feet by which the elevation of the established grade increases or decreases in a distance of one station. If the grade is ascending, the gradient is considered positive; if descending, it is negative. Use is made of the % sign in order to indicate the gradient. Thus, the gradient or a grade that rises 1 foot per station is + 1%; if the grade falls $2\frac{1}{3}$ feet in every 100 feet, its gradient is − 2.33%; etc. When the gradients are known and also the elevation of any one point of the established grade, the elevations of other points on this grade are easily computed, as will be presently explained.

15. The Depth of Center Stake.—Having set stakes on the center line at every full station and also at all intermediate points at which the inclination of the natural surface of the ground changes abruptly, the engineer should determine, by leveling, the elevation of the natural surface of the ground at each stake, and construct a profile $ABCDE$, Fig. 4, as explained in *Leveling*. The difference between the elevation of the natural surface at any stake and the elevation of the established grade at that stake is called the **depth of the stake.** The depth should be clearly marked on each stake, preceded by the letter C or F to indicate a cut or a fill.

Thus, if $ABCDE$ is the natural surface in Fig. 4, stakes will be set at the full stations A, B, M, D, and N, and also at the points K and C at which the slope of the profile of the natural surface changes. If the gradient of AG is + .76% and the elevation of A is 100.00 feet, the elevation of the point E will evidently be $100.00 + .76 \times 1 = 100.76$ feet, or (closely enough) 100.8 feet. If the elevation of the natural surface at B is 103.2, the depth of stake at B will be $103.2 - 100.8 = 2.4$ feet. This stake will therefore be marked C 2.4. The elevation of F will be $100.00 + .76 \times 2.51 = 101.9$ feet. If the elevation of C is 97.3 feet, the depth of the stake at C will be $101.9 - 97.3 = 4.6$ feet. This stake will be marked F 4.6.

Two additional columns, headed "Subgrade" and "Center Depth," must be added to the left-hand page of the level

book. In the column headed "Subgrade," the elevation of the established grade at each stake is entered; and in the column headed "Center Depth," the cut or fill is written. The gradient is usually written along the first column, as indicated in the following example. In this example, the columns of rod readings from which the elevations of points on the natural surface are obtained are omitted. The student is already familiar with the method of finding these elevations from the rod readings.

EXAMPLE.—Stakes are set at the stations indicated in the first column of the accompanying field notes. The gradient is + .76% from Sta. 90 to Sta. 93, and − .50% beyond Sta. 93. The elevation of the established grade at Sta. 90 is 100.00 feet; the elevation of the natural surface at each stake is given in the third column. To find the center depth at each stake. (See Fig. 4.)

Station	Subgrade	Elevation	Depth of Center Stake
94	101.8	102.6	C .8
93	102.28	103.3	C 1.0
92 + 51	101.9	97.3	F 4.6
92	101.5	99.6	F .9
91 + 32	101.0	104.1	C 3.1
91	100.8	103.2	C 2.4
90	100.00	100.0	0.0

SOLUTION.—The elevations of the subgrade at the station stakes are determined as follows:

STATION	ELEVATION
91	$100.00 + 1.00 \times .76 = 100.8$
91 + 32	$100.00 + 1.32 \times .76 = 101.0$
92	$100.00 + 2.00 \times .76 = 101.5$
92 + 51	$100.00 + 2.51 \times .76 = 101.9$
93	$100.00 + 3.00 \times .76 = 102.28$
94	$102.28 + 1.00 \times - .50 = 101.8$

The center depth is the difference between the corresponding numbers in the second and third columns. This is a fill if the subgrade is higher than the natural surface; otherwise, it is a cut.

EXAMPLES FOR PRACTICE

In the following examples, the elevations of the natural surface at the stakes indicated are given. It is required to find the depth at each center stake.

1. Sta. 3, 65.0; Sta. 4, 67.1; Sta. 5, 70.8; Sta. 5 + 20, 71.3; Sta. 5 + 80, 69.8; Sta. 6, 70.9. Elevation of subgrade at Sta. 3 = 66.40; gradient = + 1.3%.

Ans. F 1.4; F .6; C 1.8; C 2.0; F .2; C .6

2. Sta. 31, 134.9; Sta. 32, 133.0; Sta. 32 + 70, 132.1; Sta. 33, 132.6; Sta. 33 + 55, 139.6; Sta. 34, 132.4; Sta. 35, 129.2. Elevation of subgrade at Sta. 31 = 133.61; gradient = − 1.22%.

Ans. C 1.3; C .6; C .6; C 1.4; C 9.1; C 2.4; C .5

3. Solve example 1, if the gradient is + 2% from Sta. 3 to Sta. 5 + 20, and − .40% beyond Sta. 5 + 20, the elevation of subgrade at Sta. 3 being 65.5 feet. Ans. F .5; F .4; C 1.3; C 1.4; C .1; C 1.3

4. Solve example 2, if the gradient is − .70% from Sta. 31 to Sta. 33 and + .10% beyond Sta. 33, and if the elevation of subgrade at Sta. 31 is 134.0 feet. Ans. C .9; F .3; F .7; 0.0; C 6.9; F .3; F 3.6

16. Slope Stakes.—For the purposes of earthwork, it is necessary to know where the sloping sides ab and ce, Figs. 1 and 2, of a finished cut or fill intersect the natural surface of the ground. These points, determined as explained in the next article, are marked by stakes called **slope stakes.** The operation of locating the slope stakes is called **cross-sectioning.**

Thus, in Fig. 5, a slope stake will be driven at m and one at m'. These stakes are usually not driven vertically, but are leaned outwards from the center line. On the inner face of the stake at m, the cut mk is written; and, similarly, on the inner face of the stake at m', the cut $m'k'$ is written. These two stakes, together with the center stake at c, furnish all the information that the contractor requires to guide him in excavating the section $m\,l\,l'\,m'$.

17. To Locate the Slope Stakes.—

Let b = width $l\,l'$, Fig. 5, of the roadbed;

d = depth ce of the center stake;

s = slope ratio = $lk \div mk = l'k' \div m'k'$.

For the upper stake at m, let

x = distance mq from slope stake to center line;

$y + d$ = elevation of m above subgrade = $qc + ce = mk$.

Similarly, for the lower stake at m', let

x' = horizontal distance $m'q'$ from m' to center line;

$d - y' = m'k'$ = elevation of m' above subgrade.

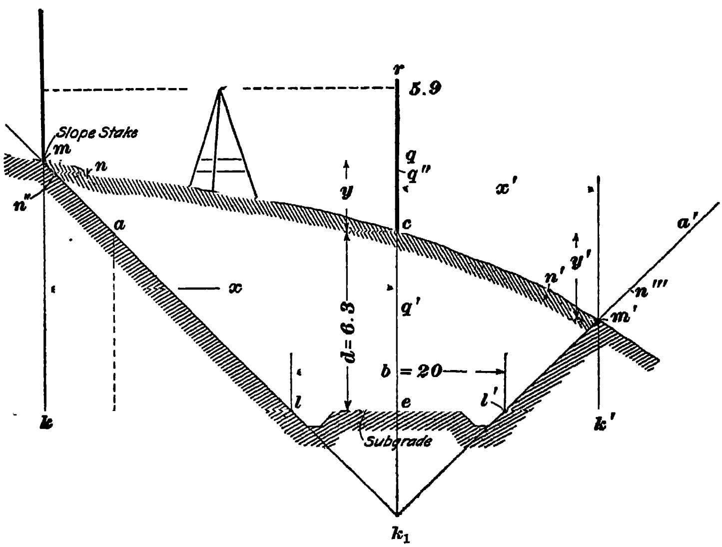

FIG. 5

Then, from the figure,

$$x = qm = ek = el + lk = \frac{b}{2} + mk \tan kml$$

$$= \frac{b}{2} + qe \tan kml = \frac{b}{2} + (d + y)\ s;$$

or,

$$x = \frac{b}{2} + sd + sy \qquad \textbf{(1)}$$

Similarly,

$$x' = \frac{b}{2} + sd - sy' \qquad \textbf{(2)}$$

If the natural surface mcm' is a level line, so that q, c, and q' are all at the same elevation, then, $y = 0$, $y' = 0$, and, therefore, by formulas **1** and **2**, and Fig. 5,

$$x = x' = ca = ca' = \frac{b}{2} + sd \qquad \textbf{(3)}$$

Formulas **1** and **2** are called **slope-stake equations**; formula **3** is called the **level-section equation.**

1. *To find the distance $m\,q = x$ from the upper slope stake to the center line.*

This distance cannot be found directly from formula **1**, because the value of the quantity y in this equation is not known until after the stake has been located. It is, therefore, found by successive trials as follows:

The required distance $m\,q$ is greater than the distance $c\,a$, and it is evident that the steeper the slope $m\,c\,m'$, the greater $m\,q$ will be. The rodman, carrying the rod and one end of the measuring tape, places the rod at some point n whose distance from c is, in his judgment, enough greater than the distance $a\,c$ computed by formula **3**, to bring the rod very near the required point m. The levelman reads the rod, and so finds the elevation of n above c; the rodman and his assistant then measure the distance from n to $c\,r$.

Let y'' be the measured elevation of n above the center stake c. Let x'' be the corresponding value of x computed by formula **1**; that is, let $x'' = \frac{b}{2} + s\,d + s\,y''$. This computed value of x'' is the distance $q''\,n''$, Fig. 5, from the center line to a point n'' of the slope whose elevation above the subgrade is equal to $y'' + d$. If the measured distance $q''\,n$ is less than the computed distance x'', the trial point n is evidently too near the center line, and the rod must be moved farther out; if the measured distance is greater than the computed distance, the rod must be moved farther in. Thus, by successive trials, a point is found for which the measured and computed values of x'' do not differ by more than .1 or .2 foot. This point will be, with sufficient approximation, the desired position of m. As an example, suppose that $d = 6.3$, and that the rod reading on c is 5.9. Suppose $s = 1.5 : 1$, and $b = 20$. Then, by formula **3**,

$$c\,a = \tfrac{20}{2} + 1.5 \times 6.3 = 19.5 \text{ feet}$$

The rodman will therefore hold the rod at some point more than 19.5 feet to the left of $c\,r$. Suppose that he holds

it 20 feet from cr, and that the reading on the rod in this position is 2.8. Then, the height of this point above c equals the reading on c minus the reading on n, or $5.9 - 2.8 = 3.1$ feet.

The computed distance from the rod to cr is

$$\frac{20}{2} + 1.5 \times 6.3 + 1.5 \times 3.1 = 24.2 \text{ feet}$$

Since the measured distance (20 feet) is much smaller than this, the rod must be moved much farther out.

Suppose that the rod is carried out 7 feet, so that the measured distance to cr is 27 feet, and suppose that the reading on the rod in this position is .8 foot. The elevation of this trial point above c will be $5.9 - .8 = 5.1$ feet, and, by formula **1**, the computed distance x'' is

$$\frac{20}{2} + 1.5 \times 6.3 + 1.5 \times 5.1 = 27.2 \text{ feet}$$

This agrees so closely with the measured distance that the slope stake may be driven at this point.

2. *To find the distance* $m'q' = x'$ *from the lower slope stake to the center line.*

The lower slope stake at m' is set in the same manner as the upper, except that the distance of each trial point below c is measured, and formula **2** is used in computing the corresponding value of x''. The distance of the trial point from cr will in this case be taken less than the distance ca' computed by formula **3**.

As in the preceding case, if the measured distance from cr to the trial point is less than the computed distance, the point should be moved out; if greater, it should be moved in. This is evident from Fig. 5; if n' is the trial point, the computed distance x'' is the distance from cr to that point n''' of the side slope whose elevation is equal to the elevation of n'. If the measured distance is less than the computed distance, the trial point is inside of the side slope k_1a', and it must therefore be moved out.

The selection of the trial point depends wholly on the judgment of the rodman. An experienced man will almost always locate the correct point at least on the second trial;

while some grow so expert that they can locate the majority of slope-stake points on the first trial.

18. Compound Sections.—Where the material to be excavated consists of a layer of earth resting on rock, the cross-section is called a **compound section.** A compound section is shown in Fig. 11. The slope ratio for the rock is less than for the earth; but, if the exact depths of the earth at the points k, a, b, and m, Fig. 11, were known, the slope stakes at k and m could be driven before any excavating had been done. As the slope of the rock and its depth below the surface are usually, however, known only very roughly, the method of setting the slope stakes in a compound section is generally as follows: The earth is first cleared away down to the rock for a width somewhat greater than that of the roadbed. Where the rock surface $a\,b$ is thus exposed, the slope stakes (or marks on the rock) at a and b are located, in the manner just described, for excavating the section $a\,d\,c\,b$. Slight **shelves** $b\,n$ and $a\,e$ are then usually cleared away on the rock to prevent in part the earth from washing into the cut. Marks are made on the rock at n and e, and finally the slope stakes m and k are set by finding by successive trials the positions of those points that satisfy the equations

$$ng = s \times gm$$
$$eg' = s \times kg'$$

in which s is the slope ratio for the earth.

EXAMPLES FOR PRACTICE

In each of the following examples, $b = 20$ feet, and $s = 1.5:1$. The letters refer to Fig. 5.

1. The depth at the center stake is 8.0 feet; the rod readings at c and n are 7.4 and 1.4, respectively. The measured distance from n to $c\,r$ is 30.5 feet. Should the trial point be moved out or in from the center line? Ans. It should be moved out slightly

2. In the preceding example, the rod reading at n' is 11.4 and the measured distance is 16.5 feet. Should the trial point be moved out or in? Ans. It should be moved in

3. The depth at the center stake is 2.0 feet; the rod reading on c is 4.6, and on n, 2.6. The measured distance from n to $c\,r$ is 18.0 feet Should the trial point be moved out or in?

Ans. It should be moved in

4. In the preceding example, the rod reading for the lower stake is 5.2, and the measured distance is 12.1 feet. Should the trial point be moved out or in? Ans. Neither

19. The Form of Notes in Cross-Section Work. When each slope stake has been set as explained in the preceding article, its distance from the center line and the elevation of the stake above or below subgrade are entered in the field book in the form of a fraction. The numerator of this fraction is the distance of the stake above or below subgrade, and the denominator is the distance of the stake from the center line. Thus, if the slope stakes in the example of Art. 17 are set at Sta. 131, the complete entry in the notebook will be as follows:

Station	Subgrade	Elevation	Center Depth	Left	Right
132	162.40	159.7	F 2.7		
131	148.80	155.1	C 6.3	$\frac{\text{C } 11.4}{27.2}$	$\frac{\text{C } 2.3}{13.5}$
130	160.40	159.8	F .6		

The first four columns have been fully explained in Art. 15. The fraction $\frac{\text{C } 11.4}{27.2}$ indicates that the left slope stake at m, Fig. 5, is 27.2 feet from the center line of the roadbed and 11.4 feet above subgrade. Similarly, the fraction $\frac{\text{C } 2.3}{13.5}$ indicates that the right slope stake m' is 13.5 feet to the right of the center line and 2.3 feet above subgrade. These expressions are called **slope-stake fractions.** It should be noticed that they are not true fractions in any sense. It is merely found convenient to write the cut or fill at each

slope stake and its measured distance from the center line in a fractional form.

Although an ordinary leveling rod may be used for setting slope stakes, the work may be done with sufficient accuracy with a light pine rod 2 inches wide, $\frac{7}{8}$ inch thick, and about 12 feet long, graduated into feet and tenths of a foot. This rod has no target, but is read directly with the telescope. It is light to carry, and if lost or damaged can easily be replaced at small cost.

COMPUTATIONS AND ESTIMATES

COMPUTATION OF VOLUME

20. Accuracy of Results Obtained.—The student should at the outset have a clear conception of the character of the work involved in earthwork surveys, and of the accuracy of the results obtainable. If material is to be excavated, it will have an upper surface *b e*, Fig. 2, that is more or less rough and irregular. Even if the excavation is made with perfectly regular slopes that form plane surfaces, yet, since the upper surface is irregular, the volume of earth cannot be exactly computed. Similarly, in a fill, since the natural surface *e b*, Fig. 1, is irregular, the volume of earth resting on *e b* cannot be found with perfect accuracy. In either case, it must be assumed that this mass of earth has a form that is practically identical with that of some geometrical solid whose volume can be exactly computed. It is true that this assumption involves some error; but the error can be reduced by taking a number of measurements sufficient to make the real volume approximate that of the assumed equivalent solid as closely as necessary. An attempt to compute the volume too closely may require an unwarranted expenditure of time and effort. Every road engineer should be able to judge what degree of accuracy is required in the surveys in order to determine the volume of the earthwork as closely as is necessary. It is never necessary to employ in the computation distances nearer than to the nearest tenth of a foot,

and it is very seldom necessary to compute volumes closer than to the nearest cubic yard.

21. Prismoids.—The definition of prismoid, the prismoidal formula, and the method of averaging end areas, with several problems illustrating the determination of volumes by both methods are given in geometry. The application of these methods to computations of earthwork will be shown in the following discussion.

If A_1 and A_2 are the areas of the bases of a prismoid; l, the perpendicular distance between them; and A_m, the area of a cross-section half way between the bases; the volumes V and V_1, as computed by the prismoidal formula and by the average end-area method, respectively, are

$$V = \frac{l}{6}(A_1 + 4\,A_m + A_2) \qquad (1)$$

$$V_1 = \frac{l}{2}(A_1 + A_2) \qquad (2)$$

The bases of the prismoid in railroad earthwork are such sections as $q\,t\,m\,p$, Fig. 7, made by vertical planes at right angles to the center line of the track. The length l of each prismoid is equal to the distance apart of the cross-sections. This is usually 100 feet, unless the surface of the ground is especially rough and irregular, when it becomes necessary to take sections at intervals of less than 100 feet. The prismoid will have four or more lateral surfaces, of which three are usually plane surfaces. The three plane surfaces are the roadbed $t\,m$, Fig. 7, which is usually a plane rectangle, and the two side slopes $t\,q$ and $m\,p$, which are usually plane surfaces in the form of trapezoids. The remaining surface $p\,q$ of the prismoid must be made to coincide with the actual surface of the ground as closely as possible.

22. Method of Calculation.—The determination of the volume by formula **2**, Art. **21**, will usually give fairly accurate results, and this method is even authorized by the laws of some American states. The prismoidal formula, however, should be used for all accurate work. This formula requires that the dimensions of the middle section whose area is A_m

shall be determined. This may be done by averaging the dimensions of the two bases and computing the area of the resulting figure. But a much simpler method is to compute the approximate volume by formula **2**, Art. **21**, and then, if necessary, to apply a correction to the result thus obtained. This correction, called the **prismoidal correction**, is the difference between the volume V computed by formula **1** and the volume V_1 computed by formula **2**; the result obtained by adding this correction to V_1 is the same as would have been obtained by a direct application of the prismoidal formula. The formula for computing the prismoidal correction will now be derived.

23. Volume and Prismoidal Correction for Triangular Prismoids.—A **triangular prismoid**, Fig. 6, is a prismoid in which the bases and all sections parallel to

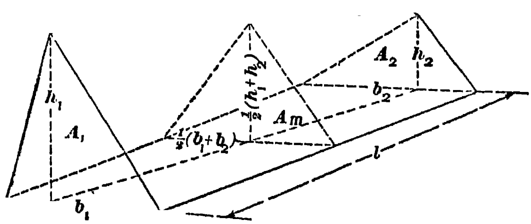

FIG. 6

them are triangles. Denoting the base and altitude of one end section by b_1 and h_1, respectively, the base and altitude of the other end section by b_2 and h_2, and the corresponding areas by A_1 and A_2, the following equations may be written:

$$A_1 = \tfrac{1}{2} b_1 h_1,\ A_2 = \tfrac{1}{2} b_2 h_2$$

The base and the altitude of the middle section, whose area is denoted by A_m, are, respectively, $\frac{1}{2}(b_1 + b_2)$ and $\frac{1}{2}(h_1 + h_2)$. Therefore,

$$A_m = \tfrac{1}{2} \times \frac{b_1 + b_2}{2} \times \frac{h_1 + h_2}{2} = \tfrac{1}{4} \times \frac{(b_1 + b_2)(h_1 + h_2)}{2}$$

Let the prismoidal correction, or the difference $V - V_1$, be denoted by C. This correction is to be added *algebraically*

to V_1 in order to obtain V. Substituting in formula **1**, Art. **21**, the values of A_1, A_2, A_m, we have

$$V = \frac{l}{6}(A_1 + 4A_m + A_2)$$

$$= \frac{l}{6}[\tfrac{1}{2}b_1 h_1 + \tfrac{1}{2}(b_1 + b_2)(h_1 + h_2) + \tfrac{1}{2}b_2 h_2]$$

$$= \frac{l}{12}(2b_1 h_1 + 2b_2 h_2 + b_1 h_2 + b_2 h_1)$$

Similarly, from formula **2**, Art. **21**,

$$V_1 = l \times \frac{A_1 + A_2}{2} = l \times \frac{\tfrac{1}{2}b_1 h_1 + \tfrac{1}{2}b_2 h_2}{2}$$

$$= \frac{l}{4}(b_1 h_1 + b_2 h_2) = \frac{l}{12}(3b_1 h_1 + 3b_2 h_2)$$

Therefore,

$$C = V - V_1$$

$$= \frac{l}{12}[2b_1 h_1 + 2b_2 h_2 + b_1 h_2 + b_2 h_1 - (3b_1 h_1 + 3b_2 h_2)]$$

$$= \frac{l}{12}(b_1 h_2 - b_1 h_1 + b_2 h_1 - b_2 h_2)$$

$$= \frac{l}{12}[b_1(h_2 - h_1) - b_2(h_2 - h_1)];$$

or, finally, $$C = \frac{l}{12}(b_1 - b_2)(h_2 - h_1) \qquad (1)$$

Also, $$V = V_1 + C \qquad (2)$$

It should be constantly borne in mind that C is to be added *algebraically* to V_1. If C is negative, this shows that the approximate volume V_1 is too large, and must be decreased; if C is positive, the approximate volume V_1 is too small, and must be increased.

A study of the correction will show that, if either the bases or the altitudes of the two end sections are equal, one of the factors $(b_1 - b_2)$ or $(h_2 - h_1)$ will become zero, and therefore the correction becomes zero. It shows also that, when one or both of these factors are small, the correction is a correspondingly small quantity; and that, when (as is usually the case) the breadth and height at one section are both smaller or both larger than the breadth and height at the other section, the correction is *negative*. Thus, if b_2 is less than b_1

and h_2 is less than h_1, then $b_1 - b_2$ is positive, $h_2 - h_1$ is negative, and, therefore, C is negative. But when C is negative, V_1 is greater than the true volume V; that is, the method of averaging end areas usually gives a result that is too large. When the difference of the breadths and heights is very large, the correction is very large, and V_1 is very greatly in error. Thus, for a pyramid, in which both b_2 and h_2 are zero, the correction is

$$\frac{l}{12}(b_1 - 0)(0 - h_1) = -\frac{b_1 h_1 l}{12}$$

The true volume is $\frac{1}{6} b_1 h_1 l$, and therefore the error in the value of V_1 is one-half, or 50 per cent., of the true volume. This extreme case shows the importance of computing the prismoidal correction when the areas of the bases are very unequal.

EXAMPLE.—The dimensions of the bases of a triangular prismoid are: $b_1 = 18$ feet, $h_1 = 8$ feet, $b_2 = 12$ feet, and $h_2 = 9$ feet. To find the volume of this prismoid, in cubic yards, if the length of the prismoid is one station.

SOLUTION.—The areas of the bases are:

$$A_1 = \tfrac{1}{2} \times 18 \times 8 = 72 \text{ sq. ft.}$$
$$A_2 = \tfrac{1}{2} \times 12 \times 9 = 54 \text{ sq. ft.}$$

Substituting these values in formula **2**, Art. **21**, and dividing by 27 to reduce to cubic yards,

$$V_1 = \tfrac{100}{2} \times (72 + 54) \div 27 = 233.33 \text{ cu. yd., nearly}$$

Substituting the given values in formula **1** above, and dividing by 27 to reduce to cubic yards,

$$C = \tfrac{100}{12} \times (18 - 12) \times (9 - 8) \div 27 = 1.85 \text{ cu. yd.}$$

Therefore, by formula **2**,

$$V = 233.33 + 1.85 = 235.18 \text{ cu. yd., or, say, 235 cu. yd. Ans.}$$

EXAMPLES FOR PRACTICE

NOTE.—Results are given to the nearest cubic yard.

1. Solve the example in Art. **23** if $b_1 = 20$ feet, $h_1 = 10$ feet, $b_2 = 10$ feet, $h_2 = 5$ feet, and $l = 1$ station. Ans. $V = 216$ cu. yd.

2. If, in Fig. 5, $l\,l' = 20$ feet and $e\,k_1 = l_s \div s = 10 \div 1.5 = 6\frac{2}{3}$ feet, and if the other base of the prismoid is an exactly equal triangle 100 feet distant, find the number of cubic yards of earth in this triangular prismoid. What is the prismoidal correction for this prismoid?
Ans. $V = 247$ cu. yd.; $C = 0$

3. If $a\,a'\,k_1$, Fig. 5, is the base of a prismoid in which $a\,a' = 39$ feet and $c\,k_1 = 13$ feet, and if, at the other base of the prismoid 100 feet distant, $a\,a' = 30$ feet and $c\,k_1 = 10$ feet, find the volume of the triangular prismoid whose bases are the figures $a\,k_1\,a'$. Ans. $V = 739$ cu. yd.

24. Three-Level Sections.—Where the surface of the ground is fairly regular, it is sufficiently accurate to determine the elevation of the center point and the distance and elevation of the two slope stakes. The method assumes that the straight lines $c\,q$ and $c\,p$, Fig. 7, that join the center with the slope stakes are on the surface of the ground. When this method is used, the sections are called **three-level sections.**

In Fig. 7, and elsewhere throughout these earthwork calculations, b represents the width $t\,m$ of the roadbed; a, the depth $k\,n$ below the roadbed to where the two side-slope lines produced intersect; d, the center depth $c\,k$; and w_l and w_r the horizontal distances from the center to the left-hand and the right-hand slope stake, respectively. The triangle below the roadbed, with base b and altitude a, is called the **grade triangle,** and its area equals $\frac{1}{2}\,a\,b$. The altitude a is found from the relation (see Art. 8)

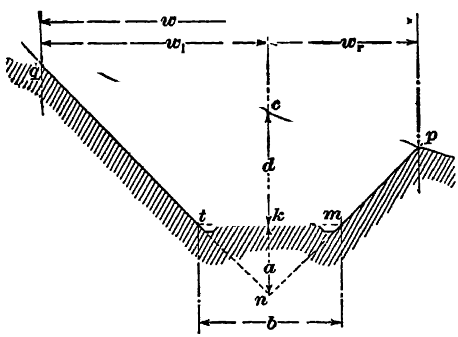

FIG. 7

$$a = n\,k = m\,k \cot m\,n\,k = \frac{m\,k}{\tan m\,n\,k} = \frac{\frac{1}{2}\,b}{s} \qquad (1)$$

If the area of the grade triangle is temporarily added to the area $t\,m\,p\,q$, the total area of the section $q\,n\,p$ may be considered as composed of the two triangles $p\,n\,c$ and $q\,n\,c$, both having the base $a + d$ and altitudes w_l and w_r, respectively. The respective areas of these triangles will therefore be $\frac{1}{2}\,(a + d)\,w_l$ and $\frac{1}{2}\,(a + d)\,w_r$. The area of the section $t\,m\,p\,q$ to be excavated is equal to the sum of the areas of these two

triangles minus the area of the grade triangle $t\,m\,n$. If the area of the section $t\,m\,p\,q$ is denoted by A_1, we shall have

$$\begin{aligned} A_1 &= \tfrac{1}{2}\,(a+d)\,(w_l+w_r) - \tfrac{1}{2}\,a\,b \\ &= \tfrac{1}{2}\,[(a+d)\,w - a\,b] \qquad (2) \end{aligned}$$

in which $w_l + w_r$ is denoted by w.

Similarly, if d' and w' denote the center depth and the total width, respectively, of the cross-section at the next stake of the center line, and A_2 denotes the area of this section, we shall have, since a and b have the same values at the two sections,

$$A_2 = \tfrac{1}{2}\,[(a+d')\,w' - a\,b] \qquad (3)$$

The approximate volume of the prismoid whose bases are the parallel sections $q\,t\,m\,p$ and whose length is the distance apart of the two sections is found by substituting the foregoing values of A_1 and A_2 in formula **2**, Art. **21**. This gives

$$\begin{aligned} V_1 &= \frac{l}{2}\,(A_1 + A_2) \\ &= \frac{l}{4}\,[(a+d)\,w + (a+d')\,w' - 2\,a\,b] \qquad \mathbf{(1)} \end{aligned}$$

If the dimensions of the two cross-sections are very nearly equal, formula **1** will give a sufficiently accurate value of the volume of the prismoid; but if the two sections differ considerably, the approximate volume V_1 must be corrected by adding to it the prismoidal correction. This correction is computed as follows: Let C' be the prismoidal correction for the prismoid whose bases are the triangles $q\,c\,n$; C'', the prismoidal correction for the prismoid whose bases are the triangles $p\,c\,n$; and C''' the prismoidal correction for the prismoid whose bases are the grade triangles $t\,n\,m$. Let, also, V_1', V_1'', and V_1''' be the approximate values of the volumes of these three prismoids, respectively, computed by formula **2**, Art. **21**; and let V', V'', and V''' be the respective values computed by the prismoidal formula. An expression for C' is found by substituting in formula **1**, Art. **23**, $d + a$ for b_1, $d' + a$ for b_2, w_l for h_1, and w_l' for h_2. This gives

$$\begin{aligned} C' &= \frac{l}{12}\,(w_l - w_l')\,[(d'+a) - (d+a)] \\ &= \frac{l}{12}\,(w_l - w_l')\,(d' - d) \end{aligned}$$

In a similar manner,

$$C'' = \frac{l}{12}(w_r - w_r')(d' - d)$$

$$C''' = 0$$

Therefore, by formula **2**, Art. **23**,

$$V' = V_1' + C';\ V'' = V_1'' + C'';\ V''' = V_1''' + C'''$$

For the required volume V we have

$$V = V' + V'' - V''' = (V_1' + V_1'' - V_1''') + (C' + C'' - C''')$$

But $\quad V_1' + V_1'' - V_1''' = V_1$, and $V = V_1 + C$

Therefore, $\quad C = C' + C'' + C'''$

or, substituting the values of C', C'', C''', and reducing,

$$C = \frac{l}{12}(w - w')(d' - d) \qquad (2)$$

When the excavation is in earth and the difference between d and d' does not exceed 3 or 4 feet, formula **1** will usually give a sufficiently accurate result. If $d' - d$ exceeds 5 feet, or if the excavation is in rock, the prismoidal correction should be computed and applied.

25. Illustrative Example.—The form in which the computation of volume should be arranged when the cross-sections are three level sections is shown on page 23. The figures in the first four columns are written while the survey is being made, as was explained in Art. **19.** The figures in columns 5, 6, and 7 are used for computing the average-end area volume V_1 by formula **1**, Art. **24**; those in columns 8, 9, and 10 are employed in computing the prismoidal correction by formula **2**, Art. **24**; and the figures in the last two columns are used for computing the correction for curvature, as will be explained in a subsequent article.

The values of V_1 for the prismoids included between the successive cross-sections are found as follows: Since the results are always expressed in cubic yards, formula **1**, Art. **24**, becomes, for the volume between two full stations ($l = 100$),

$$V_1 = \frac{100}{4\times 27}(a+d)w + \frac{100}{4\times 27}(a+d')w' - \frac{2\times 100}{4\times 27}\times a\times b \qquad \mathbf{(1)}$$

FORM OF NOTES FOR THREE-LEVEL GROUND

1	2	3	4	5	6	7		8	9	10	11	12
Station	Center Depth	Left	Right	$(a + d)$	w	Volumes		$w - w'$	$d' - d$	Prismoidal Correction C	$x_l - x_r$	Curvature Correction
						(a)	(b)					
25	C 2.4	$\frac{C\ .6}{11.9}$	$\frac{C\ 4.7}{18.0}$	9.7	29.9	269	426	+18.1	−5.7	−21	− 6.1	− 2
24 + 35	C 8.1	$\frac{C\ 5.9}{19.9}$	$\frac{C\ 11.4}{28.1}$	15.4	48.0	684	531	+16.0	−3.7	− 6	− 8.2	− 3
24	C 11.8	$\frac{C\ 8.8}{24.2}$	$\frac{C\ 19.2}{39.8}$	19.1	64.0	1,132	1,601	−14.4	+2.4	−11	−15.6	−11
23	C 9.4	$\frac{C\ 4.8}{18.2}$	$\frac{C\ 13.6}{31.4}$	16.7	49.6	767	1,048	− 3.3	+3.2	− 3	−13.2	− 7
22	C 6.2	$\frac{C\ 3.4}{16.1}$	$\frac{C\ 12.8}{30.2}$	13.5	46.3	579					−14.1	
										−41		−23

Volume by average end areas 3,606
Prismoidal correction −41

Volume by prismoidal formula 3,565

Roadbed 22 feet wide. Slope ratio = 1.5 to 1. 7° curve to the right.

If the slope is $1\frac{1}{2}:1$ and the width of the roadbed is 22 feet, we have, by equation (1), Art. **24**,

$$a = \frac{\frac{1}{2} \times 22}{\frac{3}{2}} = 7.3, \text{ for all sections}$$

The sums of the constant depth a and the variable depths d in the second column are written in the fifth column. Thus, at Sta. 22, $a + d = 7.3 + 6.2 = 13.5$ feet; at Sta. 23, $a + d = 7.3 + 9.4 = 16.7$ feet. The total width at each station is written in the sixth column. Since, in Fig. 7, $w = w_l + w_r$, and since the measured distances w_l and w_r are the denominators of the fractions in columns 3 and 4, respectively, it is only necessary to add the two denominators at each station to obtain the numbers in column 6. Thus, at Sta. 22, $w = 16.1 + 30.2 = 46.3$; at Sta. 23, $w = 18.2 + 31.4 = 49.6$ feet.

To compute the value of V_1 between Sta. 22 and Sta. 23, the proper values must be substituted in formula **1.** This gives

$$V_1 = \frac{100}{4 \times 27} \times 13.5 \times 46.3 + \frac{100}{4 \times 27} \times 16.7 \times 49.6$$
$$- \frac{2 \times 100}{4 \times 27} \times 7.3 \times 22 = 579 + 767 - 298 = 1{,}048 \text{ cu. yd.}$$

The number 579 is written in column 7 (a) opposite Sta. 22, and 767 in the same column opposite Sta. 23. The result, 1,048 cubic yards, is written opposite Sta. 23 in column 7 (b).

In a similar manner, we have, for the volume of the prismoid between Sta. 23 and Sta. 24,

$$V_1 = \frac{100}{4 \times 27} \times 16.7 \times 49.6 + \frac{100}{4 \times 27} \times 19.1 \times 64$$
$$- \frac{2 \times 100}{4 \times 27} \times 7.3 \times 22$$

The first term of this expression has already been computed, and its value, 767 cubic yards, has been written in column 7 (a) opposite Sta. 23. The last term is the constant volume 298 cubic yards. It is therefore necessary to compute the second term only. Its value is found to be 1,132 cubic yards, and this is written in column 7 (a) opposite Sta. 24. We then have

$$V_1 = 767 + 1{,}132 - 298 = 1{,}601 \text{ cubic yards,}$$

and this result is written in column 7 (b).

It is thus seen that, at each station, it is necessary to compute but one term of formula **1**; this term is the value of $\frac{100}{4 \times 27}(a + d)\,w$ for that station. The value of this term for each station is written in column 7 (a). If the stations are 100 feet apart, any number in column 7 (b) is obtained by adding the two preceding numbers in column 7 (a) and subtracting 298 cubic yards from the resulting sum. The result so obtained is the value of V_1 for a prismoid 100 feet long. But if the two stations are less than 100 feet apart, the result must be multiplied by the ratio of their distance to 100 feet to obtain the volume of the prismoid. This volume is then written in column 7 (b). For example, for the prismoid between Sta. 24 and Sta. 24 + 35, we should obtain, if the prismoid were 100 feet long,

$$1{,}132 + 684 - 298 = 1{,}518 \text{ cubic yards}$$

Since the length is but 35 feet, the actual value of V_1 is

$$\tfrac{35}{100} \times 1{,}518 = 531 \text{ cubic yards,}$$

and this number is written in column 7 (b).

It is usually more convenient to compute all the numbers in each column before passing on to the next column. When column 7 (b) has been filled up, the number of this column opposite each station is the approximate number of cubic yards, computed by average end areas, contained between that station and the preceding station. Thus, 1,048 is the approximate number of cubic yards between Sta. 23 and Sta. 22; 531 is the approximate number between Sta. 24 + 35 and Sta. 24; etc. The total approximate number of cubic yards, between Sta. 22 and Sta. 25, as computed by average end areas, is, therefore,

$$1{,}048 + 1{,}601 + 531 + 426 = 3{,}606 \text{ cubic yards}$$

The prismoidal correction must now be computed.

Since the result is to be expressed in cubic yards, formula **2**, Art. **24**, becomes

$$C = \frac{l}{12 \times 27}(w - w')(d' - d) \qquad (2)$$

The successive values of $w - w'$ in column 8 are obtained by subtracting each number in column 6 from the number

just below it in this column. Thus, for the prismoid between Sta. 22 and Sta. 23, $w = 46.3$, $w' = 49.6$; and $w - w' = -3.3$ feet. Similarly, the values of $d' - d$ in column 9 are obtained by subtracting each number in column 2 from the number just above it in this column. Thus, for the first prismoid, $d = 6.2$, $d' = 9.4$, and $d' - d = +3.2$ feet.

The numbers in column 10 are the values of the prismoidal correction computed by formula **2**. Thus, for the first prismoid, since $l = 100$,

$$C = \frac{100}{12 \times 27} \times -3.3 \times 3.2 = -3 \text{ cubic yards;}$$

for the second prismoid,

$$C = \frac{100}{12 \times 27} \times -14.4 \times 2.4 = -11 \text{ cubic yards;}$$

and similarly for the remaining prismoids.

The volume of the first prismoid, as obtained by the prismoidal formula, is, therefore, 1,048 − 3 = 1,045 cubic yards; that of the second, 1,601 − 11 = 1,590 cubic yards, etc.

EXAMPLES FOR PRACTICE

1. In the example just given, compute the volume of the prismoid between Sta. 24 and Sta. 24 + 35. Ans. $V_1 = 531$; $V = 525$ cu. yd.

2. In the foregoing example, compute the volume of the prismoid between Sta. 24 + 35 and Sta. 25. Ans. $V_1 = 426$; $V = 405$ cu. yd.

3. If the roadbed to which the accompanying notes refer is 21 feet wide and the slope is $1\frac{1}{2} : 1$, find the volume of the prismoid between Sta. 161 and Sta. 162.

Station	Center Depth	Left	Right
163	C 4.6	$\frac{\text{C } 2.4}{14.1}$	$\frac{\text{C } 0.0}{10.5}$
162	C 4.6	$\frac{\text{C } 2.1}{13.6}$	$\frac{\text{C } 4.1}{16.6}$
161	C 2.1	$\frac{\text{C } 4.0}{16.5}$	$\frac{\text{C } 6.1}{19.6}$

Ans. $V_1 = 356$; $V = 361$ cu. yd.

26. Irregular Sections.—The method of three-level sections is always sufficiently accurate for preliminary work; and, when the surface of the ground is fairly regular, it is sufficiently accurate for the final computations. When, however, the surface of the ground is very irregular, it becomes necessary, in order to obtain the volume with reasonable accuracy, to measure the distance from the center to various points in the cross-section where the slope changes, and to obtain the elevations of those points above the roadbed. This produces what is called an **irregular section,** such as is illustrated in Fig. 8. If two such sections form the bases of a prismoid, the volume can be computed by applying the prismoidal formula, or by using the method of averaging end areas, and then applying a prismoidal correction equal to the sum of the corrections for the different elementary triangular prismoids into which the irregular prismoid may be divided. While this method is theoretically applicable, and is sometimes adopted, it is usually considered a needless refinement. As a compromise between this method of extreme accuracy and the too rough method of averaging end areas, a prismoidal correction is usually computed by treating the bases, *for the purpose of this computation only*, as three-level sections. This gives only an approximate value of the correction, but a value that is generally very nearly exact.

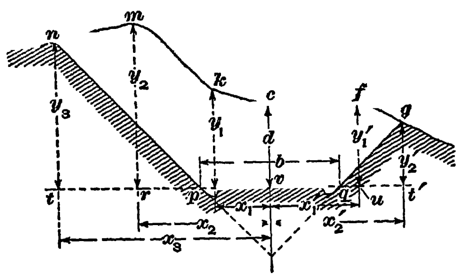

FIG. 8

The irregular section shown in Fig. 8 has two intermediate points m and k on one side of the center, and one intermediate point f on the other. The distances out and the heights of the points above the roadbed are as shown. The lines cf, fg, ck, etc. are treated as straight lines. Denoting a trapezoid by the letters in two diagonally opposite vertexes, we have, for the area A of the section $npqgfckm$,

A = trapezoid nr + trapezoid rk + trapezoid kv + trapezoid cu + trapezoid ft' − triangle ntp − triangle gqt';

or, writing the expressions for these areas in order,

$$A = \tfrac{1}{2}(y_3 + y_2)(x_3 - x_2) + \tfrac{1}{2}(y_2 + y_1)(x_2 - x_1) + \tfrac{1}{2}(y_1 + d)x_1 + \tfrac{1}{2}(y_1' + d)x_1' + \tfrac{1}{2}(y_2' + y_1')(x_2' - x_1') - \tfrac{1}{2}\left(x_3 - \frac{b}{2}\right)y_3 - \tfrac{1}{2}\left(x_2' - \frac{b}{2}\right)y_2'$$

Performing the operations indicated and rearranging the terms,

$$A = \tfrac{1}{2}(\tfrac{1}{2}by_3 + x_3y_2 + x_2y_1 + x_1d + dx_1' + y_1'x_2' + \tfrac{1}{2}by_2' - y_3x_2 - y_2x_1 - x_1'y_2')$$

This long expression for the area may be very easily formed as follows: Write the successive slope-stake fractions (Art. **19**) in order, in a horizontal row, beginning with the extreme left slope stake; and for the center stake write the fraction $\frac{d}{0}$. At the beginning and end of the row, write the fraction $\frac{0}{\frac{1}{2}b}$. Thus, the fraction for the stake at n is $\frac{y_3}{x_3}$; for the point m, it is $\frac{y_2}{x_2}$, etc.; so that the row of fractions for Fig. 8 will be as follows:

$$\frac{0}{\frac{1}{2}b} \times \frac{y_3}{x_3} \times \frac{y_2}{x_2} \times \frac{y_1}{x_1} \times \frac{d}{0} \times \frac{y_1'}{x_1'} \times \frac{y_2'}{x_2'} \times \frac{0}{\frac{1}{2}b}$$

Next, multiply each denominator by the numerator that follows it and each numerator by the denominator that follows it, giving to those products connected with full lines the plus sign, and to those connected with dotted lines the minus sign. One-half of the algebraic sum of these products will be the desired area. This is evident, since, proceeding according to the directions, we have the positive products

$$\tfrac{1}{2}by_3,\ x_3y_2,\ x_2y_1,\ x_1d,\ dx_1',\ y_1'x_2',\ \text{and } y_2'\tfrac{1}{2}b;$$

and the negative products

$$-y_3x_2,\ -y_2x_1,\ \text{and } -x_1'y_2'$$

One-half of the algebraic sum of these is identical with the second member of the formula given above.

EXAMPLE.—The following notes having been recorded at Sta. 129; it is required to find the area of the cross-section. The roadbed is 24 feet wide.

STATION	CENTER DEPTH	LEFT			RIGHT	
129	C 8.3	$\frac{C\ 12.7}{31.0}$	$\frac{C\ 16.0}{15.0}$	$\frac{C\ 12.2}{10.5}$	$\frac{C\ 4.1}{8.2}$	$\frac{C\ 6.0}{21.0}$

SOLUTION.—The series of fractions will be as follows:

$$\left(\frac{0}{12}\right)\times\frac{12.7}{31.0}\times\frac{16.0}{15.0}\times\frac{12.2}{10.5}\times\left(\frac{8.3}{0}\right)\times\frac{4.1}{8.2}\times\frac{6.0}{21.0}\times\left(\frac{0}{12}\right)$$

These are exactly as written in the field book, except that the fraction $\frac{0}{\frac{1}{2}b} = \frac{0}{12}$ is written at the beginning and end of the row; and that for the center stake the fraction $\frac{d}{0} = \frac{8.3}{0}$ is written. The double areas, computed according to the rule, are as follows:

PLUS AREAS		MINUS AREAS	
$12 \times 12.7 =$	152.4	$12.7 \times 15.0 =$	190.5
$31.0 \times 16.0 =$	496.0	$16.0 \times 10.5 =$	168.0
$15.0 \times 12.2 =$	183.0	$8.2 \times 6.0 =$	49.2
$10.5 \times 8.3 =$	87.2	Sum =	407.7
$8.3 \times 8.2 =$	68.1		
$4.1 \times 21.0 =$	86.1		
$6.0 \times 12.0 =$	72.0		
Sum =	1144.8		

The desired area is, therefore,

$$\tfrac{1}{2}(1{,}144.8 - 407.7) = 368.6 \text{ sq. ft.} \quad \text{Ans.}$$

27. In computing the area, the line of fractions in the statement of the preceding example need not be copied from the field book. It is only necessary to write in with a lead pencil the three additional fractions of the series in the solution, which are enclosed in parenthesis, and then to form the products. After a very little practice, the student will avoid writing these fractions, merely *imagining* them to be written. The full and dotted lines of series in the solution should not be drawn on the page of the field book.

This general method for irregular sections applies to all sections, no matter at how many points n, m, k, etc.,

Fig. 8, readings are taken. If preferred, it may therefore be used for three-level sections in place of the method of Art. 24.

28. Illustrative Example: Tabulation of Data and Results.—The data and results in the following example are given in tabular form, as before, but with even greater reason on account of its greater complexity; yet, the method is but an extension of the method previously used for three-level ground. Two tables are given—(A) and (B): the first is merely a copy of the field notes, which are here made separate from the computations, since both of them require considerable space. The form of the field notes should be carefully observed.

(A)

FIELD NOTES

1	2	3	4
Station	Center Cut or Fill	Left	Right
129	C 8.3	$\frac{C\ 12.7}{31.0}$ $\frac{C\ 16.0}{15.0}$ $\frac{C\ 12.2}{10.5}$	$\frac{C\ 4.1}{8.2}$ $\frac{C\ 6.0}{21.0}$
+40	C 13.2	$\frac{C\ 22.8}{46.2}$ $\frac{C\ 20.4}{31.0}$ $\frac{C\ 18.2}{19.5}$	$\frac{C\ 12.8}{13.7}$ $\frac{C\ 10.4}{27.6}$
128	C 10.9	$\frac{C\ 18.6}{39.9}$	$\frac{C\ 8.0}{4.2}$ $\frac{C\ 8.5}{21.7}$
127	C 8.6	$\frac{C\ 14.6}{33.9}$	$\frac{C\ 12.4}{30.6}$
126	C 4.2	$\frac{C\ 9.6}{26.4}$	$\frac{C\ 2.1}{15.1}$

Roadbed 24 feet wide in cut. Slope 1.5 : 1.

In column 1 of Table (A) is given the station number of the section; it should be observed that the notes run from the bottom of the page upwards. The notes are arranged in this way so that, when one stands on the line of the road looking forwards, the fractional expressions, which give for

(B)

Computation

1 Station	2 Double Plus Areas	3 Double Minus Areas	4 Cubic Yards (a)	 (b)	5 w	6 $w - w'$	7 $d' - d$	8 Prismoidal Correction
	152.4	190.5						
	496.0	168.0						
	183.0	49.2	683	1,215	52.0	+21.8	− 4.9	− 20
129	87.2							
	68.1							
	86.1							
	72.0							
	273.6	706.8						
	942.5	397.8						
	564.2	142.5	1,342	886	73.8	−12.2	+ 2.3	− 3
128 + 40	257.4							
	180.8							
	353.3							
	124.8							
	223.2	35.70						
	434.9							
128	45.8		874	1,688	61.6	+ 2.9	+ 2.3	+ 2
	173.6							
	102.0							
	175.2							
127	291.5		814	1,105	64.5	− 23.0	+ 4.4	− 31
	263.2							
	148.8							
	115.2							
126	110.9		291		41.5			
	63.4							
	25.2							

$$V_1 = 4{,}894 \qquad -52$$
$$C = -52$$
$$V = 4{,}842$$

each point the height and distance from the center, will have on the notebook approximately the same relative position as they have on the ground. Column 2 contains the center cut or fill, each number being preceded by F or C, to indicate fill or cut, respectively. The slope-stake figures for the left-hand side are always given at the extreme left of the space in column 3. The line between columns 3 and 4 may then represent the center line, and the intermediate points between the left-hand slope stake and the center are given in their order in column 3. Similarly, the points on the right side are placed in column 4; the figures for the right-hand slope stake are always placed in the extreme right-hand side of that column.

Table (B) shows the computation arranged in tabular form. In the first column are the station numbers; in the second are the double plus areas; and in the third are the double minus areas. From formula **2**, Art. **21**, the volume V_1, in cubic yards, of the prismoid between two full stations is given by the equation

$$V_1 = \frac{100}{2 \times 27} \times A_1 + \frac{100}{2 \times 27} \times A_2$$

In column 4 (*a*), opposite each station, is given the value of $\frac{100}{2 \times 27} \times A$ for that station. The sum of any two successive numbers in column 4 (*a*) is the volume V_1 of the prismoid between the corresponding sections, if this prismoid is 100 feet long; otherwise, this sum must be multiplied by the ratio of the length of the prismoid to 100 feet. The resulting volume is written in column 4 (*b*). The last four columns contain the computation of the prismoidal correction, performed as explained in Art. **25**.

To show clearly how the table is formed, the computation of the volume of the prismoid between Sta. 128 + 40 and Sta. 129 will now be given in full. To find the end area at Sta. 128 + 40, the following fractions are written:

$$\frac{0}{12.0} \quad \frac{22.8}{46.2} \quad \frac{20.4}{31.0} \quad \frac{18.2}{19.5} \quad \frac{13.2}{0} \quad \frac{12.8}{13.7} \quad \frac{10.4}{27.6} \quad \frac{0}{12.0}$$

The products of the numbers connected by full lines, 12.0×22.8, 46.2×20.4, etc., are written in column 2, and the products of those connected by dotted lines, 22.8×31.0, 20.4×19.5, etc., are written in column 3. The sum of the double plus areas is 2,696.6, and the sum of the double minus areas is 1,247.1. The area of the section is, therefore, $\frac{1}{2} \times (2{,}696.6 - 1{,}247.1) = 724.8$ square feet. The product $\frac{100}{2 \times 27} \times 724.8 = 1{,}342$ cubic yards is written in column 4 (*a*) of the table.

From the example in Art. **26** the area of the section at Sta. 129 is 368.6 square feet; the product $\frac{100}{2 \times 27} \times 368.6 = 683$ is written in column 4 (*a*) opposite Sta. 129.

If the prismoid were 100 feet long, the volume V_1 would be $683 + 1{,}342 = 2{,}025$ cubic yards. As the prismoid is but 60 feet long, the volume is $\frac{60}{100} \times 2{,}025 = 1{,}215$ cubic yards, and this number is written in column 4 (*b*) opposite Sta. 129.

The computation for the other stations is made in a similar way. It will be observed that the sections at Sta. 126 and Sta. 127 are three-level sections, and that in this case there are no minus areas.

The sum of the numbers in column 4 (*b*) is 4,894 cubic yards, and this is the volume V_1 of the prismoid between Sta. 126 and Sta. 129. The total prismoidal correction, obtained as explained in Art. **25**, is -52 cubic yards. Therefore, the final volume V is $4{,}894 - 52 = 4{,}842$ cubic yards.

EXAMPLES FOR PRACTICE

1. Find the volume of the prismoid between Sta. 127 and Sta. 128 in the example just given.
Ans. $V_1 = 1{,}688$; $C = +2$; $V = 1{,}690$ cu. yd.

2. Find the volume of the prismoid between Sta. 128 and Sta. 128 + 40 in example 1.
Ans. $V_1 = 886$; $C = -3$; $V = 883$ cu. yd.

3. Solve example 1 of Art. **25** by the method of this article.
Ans. $V_1 = 531$; $C = -6$; $V = 525$ cu. yd.

4. Having given the following field notes, find the volume of the prismoid between Sta. 21 and Sta. 22, if the roadbed is 20 feet wide:

Station	Center Depth	Left		Right	
22	C 6.5	C 4.1 / 16.1	C 5.0 / 8.0	C 2.0 / 8.0	C 7.5 / 21.2
21	C 5.3	C 6.2 / 19.3	C 6.0 / 12.0	C 4.0 / 10.0	C 8.9 / 23.3

Ans. $V_1 = 522$; $C = +2$; $V = 524$ cu. yd.

29. Areas of End Sections in Side-Hill Work. When the grade line runs pretty close to the surface along a side slope, it will usually happen that both cut and fill will be necessary in the same section. In such a case, it is frequently sufficiently accurate to consider that the section in either cut or fill is triangular. Thus, in Fig. 9, if the

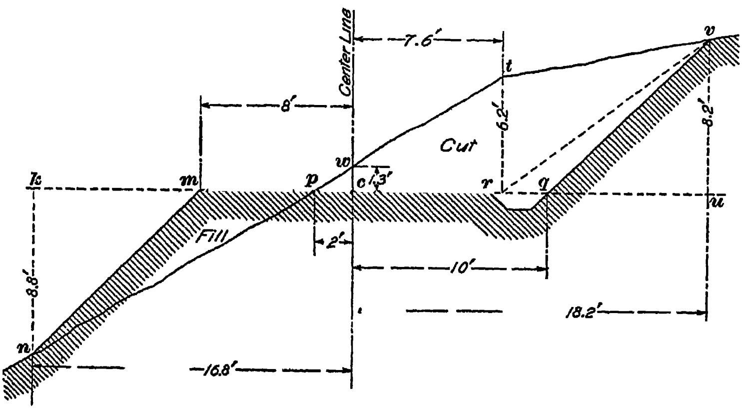

Fig. 9

slope from p to v were uniform, the area of the cut would be the area of the triangle $p\,q\,v$, and that of the fill would be the area of the triangle $p\,m\,n$. If there is an intermediate point t, however, the method of Art. 26 should be employed

to obtain the area of the section $pqvt$. Although the amount of earthwork to be done in side-hill work is usually very small, yet, as previously stated, the method of averaging end areas is generally very inaccurate, the prismoidal correction being a very large percentage of the total volume, frequently as much as one-third of the nominal volume.

As an illustration, suppose that in Fig. 9 the shoulder m of the slope is 8 feet from the center, that the fill begins at 2 feet from the center, and is a rock fill with a slope of 1 : 1; and that the slope stake n is 16.8 feet from the center. Then, $mk = ck - cm = 16.8 - 8.0 = 8.8$ feet; and, since the slope $nk \div km$ is 1 : 1, the vertical distance nk of n below subgrade will also be 8.8 feet. Assuming it to be sufficiently accurate to treat pn as a straight line, it may be considered that the section of fill is the triangle mnp, whose base mp equals $8.0 - 2.0 = 6.0$ feet and whose altitude is nk.

The area $pqvt$ is found by the formula in Art. **26**. The fraction for the point p is $\frac{0}{2}$; that for t is $\frac{C\,6.2}{7.6}$; and that for v is $\frac{C\,8.2}{18.2}$. The center depth is 1.3 feet, and the distance $cq = \frac{1}{2}b$ is 10 feet. The notes for the entire section shown in Fig. 9 will therefore be as given in the following table:

Station	Center Depth	Left		Right	
33	C 1.3	$\frac{F\,8.8}{16.8}$	$\frac{0}{2.0}$	$\frac{C\,6.2}{7.6}$	$\frac{C\,8.2}{18.2}$

The series of fractions will therefore be, considering only the section of cut,

$$\frac{0}{10} \quad \frac{0}{2} \quad \frac{1.3}{0} \quad \frac{6.2}{7.6} \quad \frac{8.2}{18.2} \quad \frac{0}{10}$$

The double areas are as follows:

	Plus Areas	Minus Areas
	2.6	62.3
	9.9	
	112.8	
	82.0	
Sum =	207.3	

The desired area for cut is, therefore,

$$\tfrac{1}{2} \times (207.3 - 62.3) = 72.5 \text{ square feet}$$

30. Computation of Volumes in Side-Hill Work. The volumes of the prismoids for cut and for fill will be computed separately. The areas of the bases of each prismoid are first found as explained in Art. **29**, and the volumes V_1 are computed by the formula

$$V_1 = \frac{l}{2}(A_1 + A_2)$$

To find the prismoidal correction, it is sufficiently accurate to regard, *for the purpose of computing this correction only*, the bases of the prismoids as triangles, and to compute the prismoidal correction by formula **1**, Art. **23.**

EXAMPLE.—It is required to compute from the following notes the volume of cut and fill, the roadbed being 20 feet wide in cuts and 16 feet wide in fills (see Fig. 9):

Station	Center Depth	Left		Right		
33	C 1.3	$\frac{\text{F } 8.8}{16.8}$	$\frac{0}{2.0}$		$\frac{\text{C } 6.2}{7.6}$	$\frac{\text{C } 8.2}{18.2}$
32	F 2.0	$\frac{\text{F } 11.4}{19.4}$		$\frac{0}{3.4}$	$\frac{\text{C } 3.3}{11.6}$	$\frac{\text{C } 6.0}{16.0}$

SOLUTION.—At Sta. 32, the distance cp = 3.4 ft., and p is on the right of c. Hence, the base pm of the triangle of fill = 8.0 + 3.4 = 11.4 ft. Since nk = 11.4 ft., the area of this triangle is $\frac{1}{2} \times 11.4 \times 11.4 = 65.0$ sq. ft., area of fill.

To compute the area of the cut, we write the series

$$\frac{0}{3.4} \times \frac{3.3}{11.6} \times \frac{6.0}{16.0} \times \frac{0}{10}$$

and obtain:

	Double Plus Areas	Double Minus Areas
	5 2.8	6 9.6
	6 0.0	1 1.2
Sum =	1 1 2.8	8 0.8

Hence, the area is $\frac{1}{2}$ (112.8 − 80.8) = 16.0 sq. ft. (cut).

For Sta. 33, we have found in the example of Art. **29**,

area of fill = 26.4 sq. ft.; area of cut = 72.5 sq. ft.

The volume V_1 of fill will therefore be

$$\frac{100}{2 \times 27} \times (26.4 + 65.0) = 169 \text{ cu. yd. (fill)}$$

and that of cut,

$$\frac{100}{2 \times 27} \times (16.0 + 72.5) = 164 \text{ cu. yd. (cut)}$$

The prismoidal correction must now be computed. By formula **1**, Art. **23**, we obtain,

$$\text{for fill, } C = \frac{100}{12 \times 27} \times (11.4 - 6.0) \times (8.8 - 11.4) = -4 \text{ cu. yd.}$$

$$\text{for cut, } C = \frac{100}{12 \times 27} \times (6.6 - 12) \times (8.2 - 6) = -4 \text{ cu. yd.}$$

The final corrected volumes are, therefore,

$$\left.\begin{array}{l}\text{for fill, } V = 169 - 4 = 165 \text{ cu. yd.}\\ \text{for cut, } V = 164 - 4 = 160 \text{ cu. yd.}\end{array}\right\}\text{Ans.}$$

EXAMPLES FOR PRACTICE

1. Draw a figure showing the cross-section at Sta. 32 in the example just given; obtain an expression for the area of the cut as in Art. **26**, and thus show that the employment of the series in the foregoing example will give the correct area.

2. If, in the foregoing example, the field notes at Sta. 34 are as given in the accompanying table, find the volumes of the prismoids of cut and fill between Sta. 33 and Sta. 34.

$$\text{Ans.}\begin{cases}\text{For fill, } V_1 = 77,\ C = 1;\ V = 78 \text{ cu. yd.}\\ \text{For cut, } V_1 = 312;\ C = -1;\ V = 311 \text{ cu. yd.}\end{cases}$$

Station	Center Depth	Left		Right
34	C 4.0	$\frac{\text{F } 10.0}{18.0}$	$\frac{0}{5}$	$\frac{\text{C } 9.4}{19.4}$

31. Transition From Cut to Fill.—At each transition from cut to fill, the cross-section of cut gradually diminishes to zero, and the fill, commencing at zero, increases to the full-sized embankment. There is, therefore, a terminal pyramid at the end of the cut, and a similar pyramid at the beginning of the fill. A cross-section should be taken where either side of the roadbed first runs out of the cut, as the point *b* in Fig. 10. The section at that point will be a triangle, and the remainder of the cut will be a pyramid

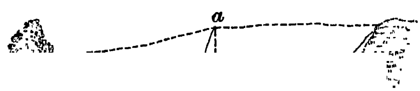

FIG. 10

whose vertex is the point *n*, where the other side of the cut reaches the surface. The initial pyramid of fill overlaps the terminal pyramid of cut for practically its whole length, the only discrepancy being that the width of the roadbed in the cut is greater than that in the fill, and therefore the apex of the pyramid of fill is not at the base of the pyramid of cut. A cross-section of the fill should be taken at the point *n*, where the roadbed attains its full width in fill. The volume of each of these terminal pyramids equals the area of its triangular base multiplied by one-third of the height, or

length. Beyond these terminal pyramids, the volume of cut or fill can be determined by the usual methods.

32. Compound Sections.—When the excavation for a cut passes partly through earth and partly through rock, it may be justifiable to use different slope ratios and different forms of cross-section, as is illustrated in Fig. 11. When estimates are being made to determine the amount of earthwork, it is sometimes required that borings shall be made to determine whether rock will be encountered before the excavation is carried to its full depth. During construction, the earth is dug away until rock is exposed; and after excavating

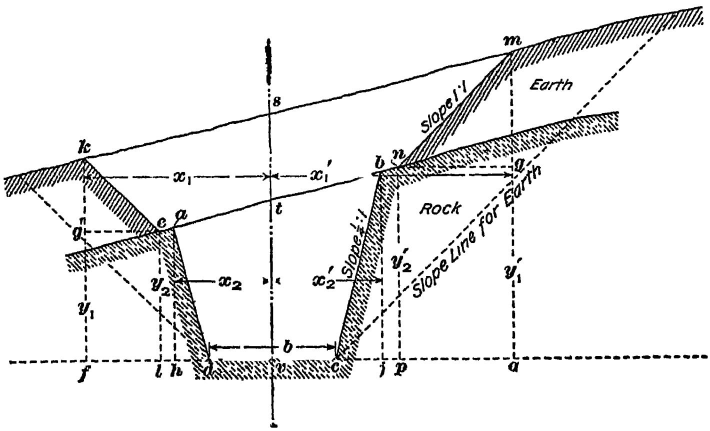

FIG. 11

into the rock to the desired depth, and using such slopes as are found necessary, the earth slopes are finally determined. Although it may be seen that there is a considerable economy so far as mere volume of excavation is concerned when rock is encountered, yet, since rock is more expensive to excavate, it frequently makes but little difference in the cost whether rock is found or not. The largely increased cross-section of earth tends to compensate for the reduced cost per cubic yard of excavating the earth.

The computation of the volume of earthwork between compound cross-sections is much more complicated than when the excavation is of only one class of material. The

area of the earthwork and the rockwork at each section must be determined separately. It is very difficult, if not impossible, to make preliminary computations, before any excavating has been done, of the precise amount of rockwork and earthwork, since they can only be made in a blind way by boring. It is necessary to know just where the rock sections "run out," and this cannot ordinarily be determined until after the excavation has been completed.

When the excavation has been completed, the areas of the end sections are determined as follows:

To find the area of $a\,b\,c\,d$, Fig. 11, of the rock section, we have, from the figure (assuming $a\,h = e\,l$, $b\,j = n\,p$, which is sufficiently close),

$$\begin{aligned}\text{area } a\,b\,c\,d &= \text{trapezoid } a\,j - \text{triangle } a\,h\,d - \text{triangle } b\,j\,c \\ &= \tfrac{1}{2}(y_2 + y_2')(x_2 + x_2') - \tfrac{1}{2}(x_2 - \tfrac{1}{2}b)y_2 \\ &\qquad - \tfrac{1}{2}(x_2' - \tfrac{1}{2}b)\,y_2' \\ &= \tfrac{1}{2}(\tfrac{1}{2}b\,y_2 + x_2 y_2' + y_2 x_2' + \tfrac{1}{2}b\,y_2') \qquad (1)\end{aligned}$$

This expression may be easily formed by a method similar to that explained in Art. **26**. Thus, if we write the series of fractions

$$\frac{0}{\frac{1}{2}b} \diagup \frac{y_2}{x_2} \times \frac{y_2'}{x_2'} \diagdown \frac{0}{\frac{1}{2}b},$$

and add the products of the quantities that are connected by full lines, the resulting sum will be the area desired.

To find the area $e\,n\,m\,k$ of the earth section, let c be the width of the **shelf** $e\,a$ or $b\,n$. Then,

$$\begin{aligned}\text{area } e\,n\,m\,k &= \text{trapezoid } k\,q - \text{trapezoid } k\,l - \text{trapezoid } l\,n \\ &\qquad - \text{trapezoid } n\,q \\ &= \tfrac{1}{2}(y_1+y_1')(x_1+x_1') - \tfrac{1}{2}(y_1+y_2)(x_1-x_2-c) \\ &\quad - \tfrac{1}{2}(y_2+y_2')(x_2+x_2'+2c) - \tfrac{1}{2}(y_1'+y_2')(x_1'-x_2'-c) \\ &= \tfrac{1}{2}\,(x_1'y_1 + y_1 x_2 + x_2' y_1' + x_1 y_1' - x_1 y_2 \\ &\quad - x_2 y_2' - y_2 x_2' - x_1' y_2') + \tfrac{1}{2}c\,(y_1 - y_2 + y_1' - y_2) \qquad (2)\end{aligned}$$

The expression in the first parenthesis is most easily formed by writing the series

$$\frac{0}{x_1'} \diagup \frac{y_1}{x_1} \times \frac{y_2}{x_2} \times \frac{y_2'}{x_2'} \times \frac{y_1'}{x_1'} \diagdown \frac{0}{x_1}$$

and forming the products indicated, giving the plus sign to those connected by the full lines, and the minus sign to those connected by the dotted lines, as explained in Art. **26.**

When the areas of the end sections of the rock and earth cuts have been found by formulas **1** and **2**, respectively, the volumes V_1 are obtained by formula **2**, Art. **21.** The prismoidal correction is then computed by formula **2**, Art. **24.** In applying this formula to the rock section, we assume

$$d = vt \text{ (Fig. 11)} = \tfrac{1}{2}(y_2 + y_2');$$

in applying it to the earth section, we assume

$$d = st = \tfrac{1}{2}(y_1 + y_1') - vt$$

Example.—From the following notes, which contain the full measurement of a portion of a completed cut, compute the volume of rock and that of earth between Sta. 362 and Sta. 363, if the roadbed is 20 feet wide, and the shelf $ae = bn$ (Fig. 11) is 1 foot wide.

Station	Left: Earth	Left: Rock	Right: Rock	Right: Earth
363	C 8.0 14.0	C 6.8 11.4	C 7.2 12.2	C 10.5 16.0
362	C 15.0 18.0	C 12.2 14.0	C 13.0 15.0	C 18.0 20.0
361	C 8.0 18.4	C 2.0 11.0	C 3.6 11.0	C 6.0 15.1

Solution.—To compute the rock area at Sta. 362, we write the series

$$\frac{0}{10} \quad \frac{12.2}{14.0} \quad \frac{13.0}{15.0} \quad \frac{0}{10}$$

From this we have

$$\text{area} = \tfrac{1}{2} \times (122 + 182 + 130 + 183) = 308.5 \text{ sq. ft.}$$

Similarly, at Sta. 363, we have the series

$$\frac{0}{10} \quad \frac{6.8}{11.4} \quad \frac{7.2}{12.2} \quad \frac{0}{10}$$

from which

$$\text{area} = \tfrac{1}{2} \times (68 + 82 + 72 + 83) = 152.5 \text{ sq. ft.}$$

Therefore, for the rock,

$$V_1 = \frac{100}{2 \times 27} \times (308.5 + 152.5) = 854 \text{ cu. yd.}$$

To find the area of the earth section at Sta. 362, we write the series

$$\frac{0}{20.0} \quad \frac{15.0}{18.0} \quad \frac{12.2}{14.0} \quad \frac{13.0}{15.0} \quad \frac{18.0}{20.0} \quad \frac{0}{18.0}$$

from which the partial area is 129.7 sq. ft. To this must be added the second term of formula **2**, or

$\frac{1}{2}c\,(y_1 - y_2 + y_1' - y_2') = \frac{1}{2} \times 1 \times (15 - 12.2 + 18 - 13) = 3.9$ sq. ft.

The total area of the earth section at Sta. 362 is, therefore, $129.7 + 3.9 = 133.6$ sq. ft.

To find the area of the earth section at Sta. 363, we write the series

$$\frac{0}{16.0} \quad \frac{8.0}{14.0} \quad \frac{6.8}{11.4} \quad \frac{7.2}{12.2} \quad \frac{10.5}{16.0} \quad \frac{0}{14.0}$$

from which the partial area is 59.4 sq. ft. The term

$\frac{1}{2}c\,(v_1 - y_2 + y_1' - y_2') = \frac{1}{2} \times 1 \times (8 - 6.8 + 10.5 - 7.2) = 2.3$ sq. ft.

Therefore, the total area is $59.4 + 2.3 = 61.7$ sq. ft. Therefore, for the earth,

$$V_1 = \frac{100}{2 \times 27} \times (133.6 + 61.7) = 362 \text{ cu. yd.}$$

To apply the prismoidal correction, we have, for the rock,

$$\begin{aligned} w &= 14.0 + 15.0 = 29.0 \\ w' &= 11.4 + 12.2 = 23.6 \\ d &= \tfrac{1}{2} \times (12.2 + 13.0) = 12.6 \\ d' &= \tfrac{1}{2} \times (6.8 + 7.2) = 7.0 \end{aligned}$$

Therefore, by formula **2**, Art. **24**,

$$C = \frac{100}{12 \times 27} \times (29.0 - 23.6) \times (7.0 - 12.6) = -9 \text{ cu. yd.}$$

Therefore, for the rock,

$$V = 854 - 9 = 845 \text{ cu. yd.} \quad \text{Ans.}$$

In the earth prismoid, we have

$$\begin{aligned} w &= 18.0 + 20.0 = 38.0 \\ w' &= 14.0 + 16.0 = 30.0 \\ d &= \tfrac{1}{2} \times (15.0 + 18.0) - 12.6 = 3.9 \\ d' &= \tfrac{1}{2} \times (8.0 + 10.5) - 7.0 = 2.3 \end{aligned}$$

Therefore, by formula **2**, Art. **24**,

$$C = \frac{100}{12 \times 27} \times (38.0 - 30.0) \times (2.3 - 3.9) = -4 \text{ cu. yd}$$

Therefore, for the earth,

$$V = 362 - 4 = 358 \text{ cu. yd.} \quad \text{Ans.}$$

33. Borrow Pits.—The name **borrow pit** is applied to an excavation made solely for the purpose of obtaining material with which to make a fill. Sometimes, a borrow pit

is made by widening a cut, as illustrated in Fig. 12, with the idea that the added width, properly graded beside the regular roadbed, may ultimately prove of use as a place for side

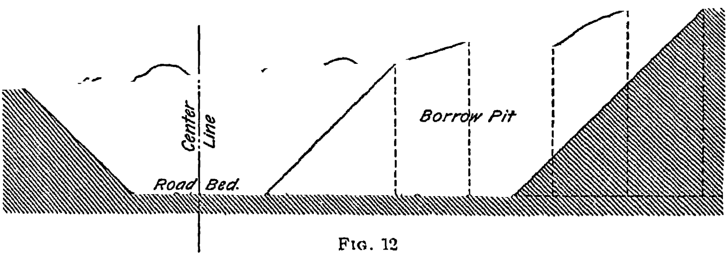

FIG. 12

tracks. In any case, since payment for earthwork is invariably made on the basis of the amount of earth excavated, the excavation is made on regular lines, so that it may be readily measured. Cross-sections of the borrow pits should

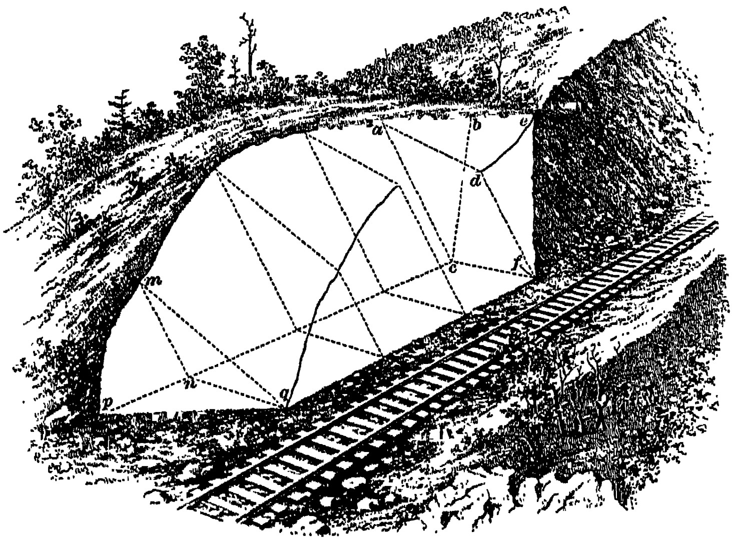

FIG. 13

be made at regular intervals, and the amount excavated should be computed by the method of average end areas, applying the prismoidal correction when necessary. In

Fig. 13 is given the perspective view of a cut with a borrow pit adjoining it, the borrow pit being made by widening the cut. The cross-sections that would be taken are indicated by the dotted lines. It should be noted that at one end the borrow pit runs to the natural end of the cut, and the terminal solid at this end is the pyramid *m n p q*. At the other end, the terminal solid is the wedge *a b c d e f*, but this occurs because the cut has been left with proper slopes. The volumes of these terminal pyramids and wedges are best obtained by taking cross-sections at their bases *m n q* and *a c f d*, and then computing their volumes by the regular rules of geometry.

34. Allowance for Ditches.—In all the computations of volumes described in the preceding articles, the sections for cuts have been taken as the figures *a c e b*, Fig. 2, and for fill as the figures *a c e b*, Fig. 1. As explained in Art. **12**, however, there will always be the two ditches *a* and *c*, Fig. 2, to be allowed for in cuts, and usually the extra ditch *b*, Fig. 2; while sometimes the ditch *e*, Fig. 1, is added to fills. The areas of the end sections in cuts may be increased by the area of the ditches; and the volumes computed by formula **2**, Art. **21**, and formula **2**, Art. **25**; but it is generally more convenient to compute the volume of the ditches separately, since each ditch is a prism.

If A is the area, in square feet, of a cross-section of a ditch, and l is the length of the ditch, in feet, the volume V_d, in cubic yards, is given by the formula

$$V_d = \frac{A\,l}{27}$$

The material excavated from the ditches *a* and *c*, Fig. 2, is available for embankment, but that excavated from the ditch *b* is usually piled on the ground between the ditch and the point *b*.

EXAMPLE.—If the two ditches *a* and *c*, Fig. 2, are 4 feet wide at the top, 1 foot wide at the bottom, and 2 feet deep, what is the additional excavation required for each 100 feet of cut on account of these ditches?

SOLUTION.—The end section of each ditch is a trapezoid whose bases are 4 ft. and 1 ft., respectively, and whose altitude is 2 ft. Therefore,

$$A = \frac{1}{2} \times (4 + 1) \times 2 = 5 \text{ sq. ft., and } V_d = \frac{5 \times 100}{27} = 18.5 \text{ cu. yd.}$$

Since there are two ditches, the volume obtained for the prismoid between any two successive full stations in cuts must be increased by

$$2 \times 18.5 = 37 \text{ cu. yd. Ans.}$$

CORRECTION FOR CURVATURE: GENERAL CASE

35. Eccentricity of a Cross-Section.—All the previous calculations have been made on the assumption that the prismoid has a straight axis and that the end planes are parallel to each other. A large proportion of railroad track is curved, and, since the successive cross-sections of the roadbed are perpendicular to the center line of the road, they are not

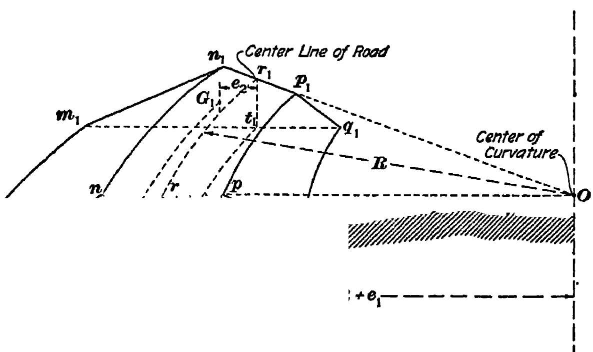

FIG. 14

parallel to each other. In Fig. 14, let A_1 be the area of the cross-section $m\,n\,p\,q$, and A_2 the area of the cross-section $m_1\,n_1\,p_1\,q_1$. Let $r\,r_1$ be the curved center line of the roadbed, and O the center of this circular curve. Let G be the center of gravity of the section $m\,n\,p\,q$, and e_1 the horizontal distance from G to the center of the roadbed. Similarly, let G_1 be the center of gravity of the section $m_1\,n_1\,p_1\,q_1$, and e_2 the horizontal distance from G_1 to the center of the roadbed. The horizontal distance from the center of gravity of any section to the center of the roadbed is called the

eccentricity of the section. Thus, e_1 is the eccentricity of the section $m\,n\,p\,q$, and e_2 is the eccentricity of the section $m_1\,n_1\,p_1\,q_1$.

36. Curvature Correction.—It may be shown by advanced mathematics that the true volume of the curved solid bounded by the sections $m\,n\,p\,q$, $m_1\,n_1\,p_1\,q_1$, Fig. 14, is *greater* than the volume computed as if the track were straight, whenever G and G_1 lie on the *outside* of the center of the roadbed, as shown in Fig. 14, and that, when G and G_1 line on the *inside* of the curved track, the volume is *less* than the volume computed as if the track were straight. The difference between the actual volume and the volume obtained by applying the prismoidal formula is called the **correction for curvature.**

Let V_c = volume of curved solid;

V = volume computed by methods already given, assuming track to be straight;

R = radius $O\,r$ of center line of roadbed, Fig. 14;

C_c = correction for curvature;

l = distance between end sections measured along center line.

Then, as can be shown by advanced mathematics,

$$C_c = \frac{l}{2R}(A_1 e_1 + A_2 e_2)$$

If the centers of gravity of the end sections lie on the *outside* of the curved center line of the roadbed, V_c is *greater* than V. If the centers of gravity of the end sections lie on the *inside* of the curved center line of the roadbed, V_c is less than V.

The expression for C_c shows that the larger the eccentricities of the end sections, the larger C_c will be, and that, if the radius of the curve is very large, C_c will be very small. For curves of very large radius, the correction is usually so small that it may be neglected. When the area of that part $r\,p\,q\,t$ of the end section that lies on the inside of the center of the track is approximately equal to the portion of the area $r\,t\,m\,n$ lying outside of the center, the eccentricity

is small, and the correction may usually be neglected, even with curves of short radii. But when the eccentricity is large (as is usually the case in side-hill work), the curvature correction may be a very considerable percentage of the volume, and should not be neglected, especially if the radius of the curve is small.

37. Eccentricity of the Center of Gravity.—As explained in Art. **35**, the eccentricity of a section is the horizontal distance of the center of gravity of the section from a vertical axis through the center of the track. A simple method for finding this distance is as follows: The section is first divided into separate parts, usually triangles, whose centers of gravity can be easily determined and their areas computed. The area of each part is then multiplied by the distance of its center of gravity from the axis; the products are added algebraically, and the result is divided by the total area of the section. The quotient is the required eccentricity. It should be borne in mind that, if distances measured from the axis in one direction are treated as positive, those measured in the opposite direction should be treated as negative. The summation of the products is always an algebraic summation. The side of the axis that has the numerically larger products is the side on which the true center of gravity lies.

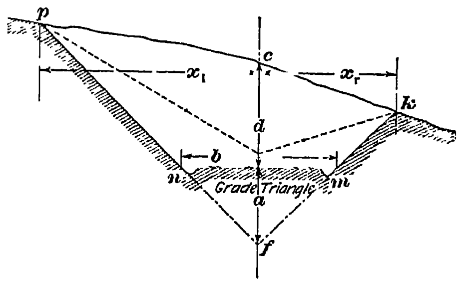

FIG. 15

Fig. 15 represents an ordinary three-level section. If the grade triangle is included, the section may be divided into two triangles *p f c* and *c f k* lying on opposite sides of the center line *c f*. The center of gravity of any triangle is always found on the line joining any vertex with the middle of the opposite side, at one-third of the distance from that side to that vertex.. In this case, the perpendicular distance

of the center of gravity of the triangle $c p f$ from the axis $c f$ is evidently one-third of x_l. The area of that triangle is $\frac{1}{2}(a+d)x_l$, and the product of the area and the distance of the center of gravity from the chosen axis is $\frac{(a+d)x_l}{2} \times \frac{x_l}{3}$. Similarly, the distance of the center of gravity of the triangle $c k f$ from the axis is $\frac{x_r}{3}$, and the product of the area by that distance is an expression similar to that already given, and with a minus sign, since the triangle is on the opposite side of the axis. Then, for the eccentricity e, we have,

$$e = \frac{\frac{(a+d)x_l}{2} \times \frac{x_l}{3} - \frac{(a+d)x_r}{2} \times \frac{x_r}{3}}{\frac{(a+d)x_l}{2} + \frac{(a+d)x_r}{2}}$$

$$= \frac{1}{3} \times \frac{x_l^2 - x_r^2}{x_l + x_r} = \frac{1}{3}(x_l - x_r)$$

The value of e given by this formula is the eccentricity of the whole triangle $p f k$. The eccentricity of the actual section $p n m k$ is somewhat greater than this; in other words, the center of gravity of the actual section is farther from the center line $c f$ than the center of gravity of the actual section and the grade triangle taken together. This is evident, since the center of gravity of the grade triangle lies on the axis $c f$.

Since the values of e computed by the foregoing formula are too small, these values, when substituted in the formula of Art. **36**, will give too small a value for the curvature correction. It is found practically that this error can be almost exactly counterbalanced by increasing the end areas A_1 and A_2 in this formula by the area of the grade triangle $n f m$.

38. Another Expression for the Curvature Correction.—If we denote by x_{l_1} and x_{r_1}, x_{l_2} and x_{r_2} the values of x_l and x_r, respectively, at two successive sections, and

denote the area of the grade triangle by T, we shall have, from the formula in Art. **37**,

$$e_1 = \tfrac{1}{3}(x_{l_1} - x_{r_1})$$
$$e_2 = \tfrac{1}{3}(x_{l_2} - x_{r_2})$$

Substituting these values of e_1 and e_2 in the formula of Art. **36**, and replacing A_1 by $A_1 + T$ and A_2 by $A_2 + T$, we obtain

$$C_c = \frac{l}{6R}[(A_1 + T)(x_{l_1} - x_{r_1}) + (A_2 + T)(x_{l_2} - x_{r_2})] \quad (1)$$

It will be observed that this formula would give an exactly correct value of C_c if the cross-sections were the triangles pfk, Fig. 15. Since the true sections are not triangles, the application of the formula introduces two errors: first, the resulting values of the areas $A_1 + T$ and $A_2 + T$ are too great; and, second, the values of the eccentricities e_1 and e_2 are too small. These two errors very nearly neutralize each other. It is not difficult to obtain the true values of e_1 and e_2 for the actual sections $m\,n\,p\,q$, by dividing these sections into triangles; but the resulting equation is not only too long to be of practical value, but in any application it would also be found to give a value of C_c differing from that obtained by the foregoing formula by only a fraction of a cubic yard. Such an error is of no consequence in earthwork computations.

For the purposes of computation, it is convenient to write formula **1** in the following form, the correction being expressed in cubic yards:

$$C_c = \frac{1}{3R}\left[\left(\frac{l}{2\times 27}\times A_1 + \frac{l}{2\times 27}\times T\right)(x_{l_1} - x_{r_1})\right.$$
$$\left.+\left(\frac{l}{2\times 27}\times A_2 + \frac{l}{2\times 27}\times T\right)(x_{l_2} - x_{r_2})\right] \quad (2)$$

EXAMPLE 1.—In the example of Art. **25**, to find the correction for curvature between Sta. 22 and Sta. 23 if the curve is a 7° curve to the right.

SOLUTION.—At Sta. 22, $x_{l_1} = 16.1$, $x_{r_1} = 30.2$, and hence $x_{l_1} - x_{r_1} = 16.1 - 30.2 = -14.1$. At Sta. 23, $x_{l_2} = 18.2$, $x_{r_2} = 31.4$, and hence $x_{l_2} - x_{r_2} = 18.2 - 31.4 = -13.2$. The values of $\frac{100}{2\times 27}\times A_1 + \frac{100}{2\times 27}\times T$

and $\frac{100}{2 \times 27} \times A_2 + \frac{100}{2 \times 27} \times T$ have been computed and tabulated in column 7 (*a*) of the table given in Art. **25.** Thus,

$$\frac{100}{2 \times 27} \times A_1 + \frac{100}{2 \times 27} \times T = 579,$$

and

$$\frac{100}{2 \times 27} \times A_2 + \frac{100}{2 \times 27} \times T = 767$$

Substituting all of these values in formula **2,** and also the value $R = 819$ ft. for a 7° curve, we obtain,

$$C_c = \frac{1}{3 \times 819} \times (579 \times -14.1 + 767 \times -13.2) = -7 \text{ cu. yd.}$$

Since x_{l1} and x_{l2} are smaller, respectively, than x_{r1} and x_{r2}, the centers of gravity of the sections lie on the right of the center line of the roadbed; since the curve turns to the right, these points therefore lie inside of the center line, and the actual volume V_c is less than V (Art. **36**). We therefore have, since the volume computed by the prismoidal formula is 1,048 − 3 = 1,045 cu. yd. (Art. **25**),

$$V_c = 1{,}045 - 7 = 1{,}038 \text{ cu. yd. Ans.}$$

EXAMPLE 2.—To find the correction for curvature between Sta. 24 and Sta. 24 + 35 in the example of Art. **25,** if the curve is a 7° curve to the right.

SOLUTION.—At Sta. 24, $x_{l1} - x_{r1} = 24.2 - 39.8 = -15.6$; and at Sta. 24 + 35, $x_{l2} - x_{r2} = 19.9 - 28.1 = -8.2$. The values of $\frac{100}{2 \times 27} \times A_1 + \frac{100}{2 \times 27} \times T$ and $\frac{100}{2 \times 27} \times A_2 + \frac{100}{2 \times 27} \times T$ given in column 7 (*a*) of the table must be multiplied by $\frac{35}{100}$, since the distance between the sections is but 35 ft. Then,

$$C_c = \frac{1}{3 \times 819} \times (\tfrac{35}{100} \times 1{,}132 \times -15.6 + \tfrac{35}{100} \times 684 \times -8.2)$$
$$= -3 \text{ cu. yd.}$$

As in example 1, the actual volume is less than the volume V computed by the prismoidal formula. From Art. **25,** $V = 531 - 6 = 525$ cu. yd.; hence,

$$V_c = 525 - 3 = 522 \text{ cu. yd. Ans.}$$

EXAMPLES FOR PRACTICE

1. In the preceding examples of this article, find the correction for curvature and the volume between Sta. 23 and Sta. 24.

Ans. $C_c = -11$; $V_c = 1{,}579$ cu. yd.

2. In the preceding examples of this article, find the correction for curvature and the volume between Sta. 24 + 35 and Sta. 25.

Ans. $C_c = -2$; $V_c = 403$ cu. yd.

CORRECTION FOR CURVATURE: SIDE-HILL WORK

39. Importance of Curvature Correction in This Case.—Side-hill sections usually have their centers of gravity at such distances from the center of the road that the correction for curvature is a large percentage of the total volume, even when that volume is small. Therefore, in side-hill work, it is nearly always necessary to compute the curvature correction. It will always be sufficiently accurate, *for the purpose of this correction*, to consider that the side-hill section is a triangle. By this means, the calculation of the position of the center of gravity is readily performed. The curvature correction for the cut and that for the fill must be computed separately.

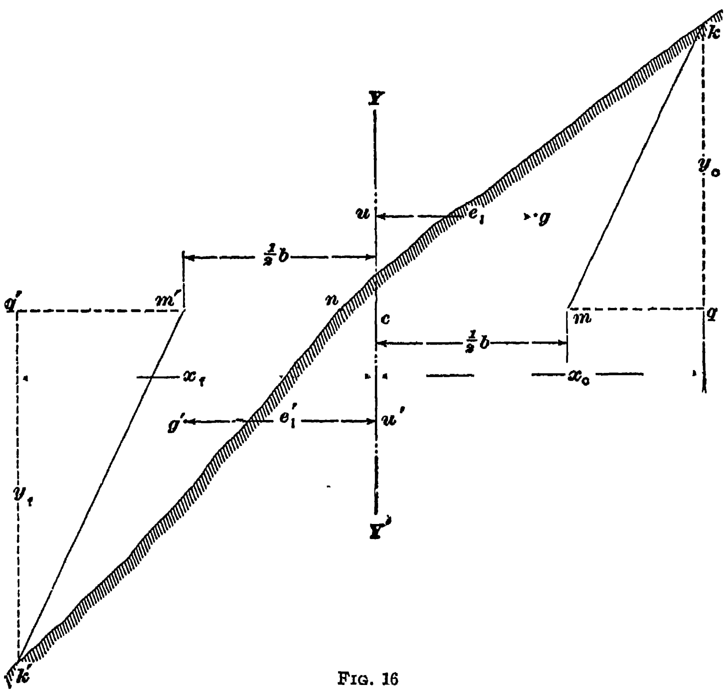

FIG. 16

40. Curvature Correction for the Cut.—In computing the curvature correction for a cut, it is necessary to distinguish two cases, as follows:

Case I.—*The center stake lies in the cut.* In Fig. 16, let the center stake c lie in the cut $k\,n\,m$. Let $c\,q = x_c$, and $q\,k = y_c$. As usual, the width of subgrade is denoted by b; but it should be observed that the value of b, and therefore of $\frac{1}{2}b$, is not the same for cuts as for fills. Let g be the center of gravity of the triangle $m\,n\,k$, and e_1 the eccentricity $g\,u$. It can be shown that if the distances of the vertexes of a triangle from a line are denoted by z_1, z_2, and z_3, the distance z_g of the center of gravity of the triangle from the same line is given by the formula

$$z_g = \tfrac{1}{3}(z_1 + z_2 + z_3)$$

it being understood that the addition is algebraic, and that distances on opposite sides of the line of reference must have

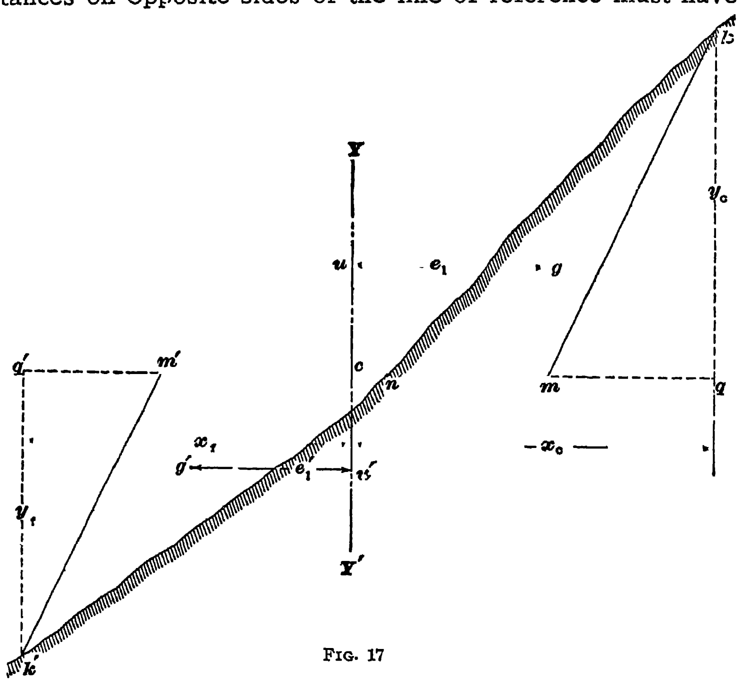

FIG. 17

opposite signs. Applying this formula to the triangle $m\,n\,k$ considering distances to the right of $Y'\,Y$ as positive, we have

$$e_1 = \tfrac{1}{3}(c\,q + c\,m - c\,n)$$

that is,

$$e_1 = \tfrac{1}{3}(x_c + \tfrac{1}{2}b - n\,c) \qquad \textbf{(1)}$$

If A_1 = area $m\,n\,k$, we have

$$A_1 = \tfrac{1}{2}\,m\,n \times q\,k = \tfrac{1}{2}(n\,c + \tfrac{1}{2}\,b) \times y_c \qquad \textbf{(2)}$$

Formulas **1** and **2** give the values of A and e to be substituted in the formula of Art. **36,** in order to obtain the correction for curvature.

Case II.—*The center stake lies in the fill.* From Fig. 17, we have, since here all the vertexes are on the right of $Y'\,Y$,

$$e_1 = \tfrac{1}{3}(x_c + \tfrac{1}{2}\,b + c\,n) \qquad \textbf{(3)}$$

Also, $$A_1 = \tfrac{1}{2}\,m\,n \times q\,k = \tfrac{1}{2}(\tfrac{1}{2}\,b - c\,n)\,y_c \qquad \textbf{(4)}$$

EXAMPLE.—If the curve to which the following notes refer is a 9° curve to the right, what is the correction for curvature for the cut between Sta. 32 and Sta. 33, the roadbed being 24 feet wide in cuts and 16 feet wide in fills?

Station	Center Depth	Left		Right	
33	F 1.3	$\frac{\text{F } 18.8}{36.2}$		$\frac{0}{3.4}$	$\frac{\text{C } 17.1}{37.7}$
32	C 2.0	$\frac{\text{F } 15.0}{30.5}$	$\frac{0}{4.4}$		$\frac{\text{C } 14.9}{34.4}$

SOLUTION.—At Sta. 32, there is a cut at the center stake. Formulas **1** and **2** are therefore applied. We have, from the notes, $x_c = 34.4$, and $c\,n = 4.4$; also, $\frac{1}{2}\,b = \frac{24}{2} = 12$. Therefore, by formula **1,**

$$e_1 = \tfrac{1}{3} \times (34.4 + 12 - 4.4) = 14 \text{ ft.}$$

By formula **2,** $A_1 = \frac{1}{2} \times (4.4 + 12) \times 14.9$ sq. ft.

At Sta. 33, there is a fill at the center stake; formulas **3** and **4** are therefore applied. At this station, $x_c = 37.7$, $\frac{1}{2}\,b = 12.0$, and $c\,n = 3.4$. Therefore, by formula **3,**

$$e_2 = \tfrac{1}{3} \times (37.7 + 12.0 + 3.4) = 17.7 \text{ ft.}$$

By formula **4,** $A_2 = \frac{1}{2} \times (12 - 3.4) \times 17.1$ sq. ft.

Substituting these values, and also $l = 100$ and $R = 637$ in formula of Art. **36,** and dividing by 27 to reduce the result to cubic yards,

$$C_c = \frac{100}{2 \times 27 \times 637} \times (\tfrac{1}{2} \times 16.4 \times 14.9 \times 14.0 + \tfrac{1}{2} \times 8.6 \times 17.1 \times 17.7)$$
$$= 9 \text{ cu. yd.}$$

Since the curve turns to the right, the center of gravity evidently lies inside of the center of the roadbed. Hence, by Art. **36,** the volume obtained by applying the prismoidal formula should be *diminished* by 9 cu. yd., in order to obtain the actual volume.

41. Curvature Correction for the Fill.—As in the case of the cut, so in computing the curvature correction for the fill, two cases are distinguished:

Case I.—*The center stake lies in the cut.* If g', Fig. 16, is the center of gravity of the triangle $k'\,m'\,n$, we shall have, denoting $c\,q'$ by x_f,

$$e_1' = g'\,u' = \tfrac{1}{3}\,(x_f + \tfrac{1}{2}\,b + c\,n) \qquad \textbf{(1)}$$

If A_1' = area $k'\,m'\,n$, and $k'\,q' = y_f$, then,

$$A_1' = \tfrac{1}{2}\,m'\,n \times k'\,q' = \tfrac{1}{2}\,(\tfrac{1}{2}\,b - c\,n)\,y_f \qquad \textbf{(2)}$$

Case II.—*The center stake lies in the fill.* In Fig. 17, the notation being as before, we have

$$e_1' = g'\,u' = \tfrac{1}{3}\,(x_f + \tfrac{1}{2}\,b - n\,c) \qquad \textbf{(3)}$$

If A_1' is the area of the fill triangle, then,

$$A_1' = \tfrac{1}{2}\,m'\,n \times k'\,q' = \tfrac{1}{2}\,(\tfrac{1}{2}\,b + c\,n)\,y_f \qquad \textbf{(4)}$$

EXAMPLE.—To find the correction for curvature for the fill in the example of Art. **40**.

SOLUTION.—For Sta. 32, formulas **1** and **2** must be used. From the notes, $x_f = 30.5$, $\tfrac{1}{2}b = 8$, and $n\,c = 4.4$. Therefore,

$$e_1' = \tfrac{1}{3} \times (30.5 + 8.0 + 4.4) = 14.3$$
$$A_1' = \tfrac{1}{2} \times (8.0 - 4.4) \times 15.0 = \tfrac{1}{2} \times 3.6 \times 15.0$$

For Sta. 33, formulas **3** and **4** must be used. Here, $x_f = 36.2$, $\tfrac{1}{2}b = 8$, and $c\,n = 3.4$; therefore,

$$e_2' = \tfrac{1}{3} \times (36.2 + 8.0 - 3.4) = 13.6$$
$$A_2' = \tfrac{1}{2} \times (8 + 3.4) \times 18.8 = \tfrac{1}{2} \times 11.4 \times 18.8$$

Hence,

$$C_c = \frac{100}{2 \times 27 \times 63\overline{7}} \times (\tfrac{1}{2} \times 3.6 \times 15.0 \times 14.3 + \tfrac{1}{2} \times 11.4 \times 18.8 \times 13.6)$$
$$= 5 \text{ cu. yd.}$$

Since the center of gravity of the fill evidently lies outside of the center line of the curve, the volume computed by the prismoidal formula should be increased by 5 cu. yd., in order to obtain the actual volume.

EXAMPLES FOR PRACTICE

1. The following notes apply to a 10° curve to the left. The roadbed is 24 feet wide in the cuts, and 16 feet wide in the fills. Find the correction for curvature for the cut between Sta. 161 and Sta. 162.

Station	Center Depth	Left		Right	
162	C 3.0	$\frac{\text{F } 25.5}{46.3}$	$\frac{0}{1.5}$		$\frac{\text{C } 22.4}{45.6}$
161	F 2.4	$\frac{\text{F } 28.5}{50.8}$		$\frac{0}{6.3}$	$\frac{\text{C } 12.2}{30.3}$

Ans. C_c = 11 cu. yd. (to be added)

2. Find the correction for curvature for the fill in the preceding example. Ans. C_c = 16 cu. yd. (to be subtracted)

3. If the roadbed to which the following notes refer is 20 feet wide in the cuts, find the correction for curvature for the cut between Sta. 22 + 40 and Sta. 23, the curve being a 10° curve to the left.

Station	Center Depth	Left		Right	
23	C 6.0	$\frac{\text{F } 18.9}{36.4}$	$\frac{0}{6.0}$		$\frac{\text{C } 20.0}{40.0}$
22 + 40	F 0.8	$\frac{\text{F } 18.0}{35.0}$		$\frac{0}{2.0}$	$\frac{\text{C } 18.0}{37.0}$

Ans. C_c = 6.8 cu. yd. (to be added)

SHRINKAGE OF EARTHWORK

42. It is usually observed that, when earth is excavated and formed into an embankment, the volume of the embankment is at first greater than that of the original excavation, but, after some time, the embankment shrinks to a volume less than that of the original excavation. This **shrinkage** generally amounts, on an average, to about 10 per cent., although in the case of very loose vegetable soil it may amount to as much as 25 per cent. Table I contains in the second column the approximate number of cubic yards of

embankment that can be formed from 1,000 cubic yards of excavation. In the third column is given the number of cubic yards of excavation required for each 1,000 cubic yards of embankment, and in the fourth column is given the per cent. of shrinkage for the various kinds of soils.

TABLE I

SHRINKAGE OF EARTHWORK

Character of Material	Embankment Obtained From 1,000 Cubic Yards of Excavation Cubic Yards	Excavation Required for 1,000 Cubic Yards of Embankment Cubic Yards	Shrinkage Per Cent.
Sand and gravel	920	1,087	8
Clay	900	1,111	10
Loam	880	1,136	12
Wet soil	850	1,200	15

EXAMPLE 1.—The volume of a cut through clay soil is 1,630 cubic yards. How many cubic yards will be contained in an embankment made from this material?

SOLUTION 1.—Since, from Table I, 1,000 cu. yd. of excavation will form only 900 cu. yd. of embankment, the desired result will be

$$\frac{900}{1000} \times 1,630 = 1,467 \text{ cu. yd.} \quad \text{Ans.}$$

SOLUTION 2.—Since, from Table I, the shrinkage is 10 per cent., the shrinkage will be 10 per cent. of 1,630 cu. yd., or 163 cu. yd. Therefore, the volume of the embankment will be

$$1,630 - 163 = 1,467 \text{ cu. yd.} \quad \text{Ans.}$$

EXAMPLE 2.—How many cubic yards of excavation in gravel are required to form an embankment of 2,200 cubic yards?

SOLUTION.—Since, from Table I, each 1,000 cu. yd. of embankment will require 1,087 cu. yd. of excavation, the desired result will be

$$\frac{1087}{1000} \times 2,200 = 2,391 \text{ cu. yd.} \quad \text{Ans.}$$

43. Growth of Rock.—When a rock excavation is formed into an embankment, it will have a volume from 40 to 80 per cent. larger than its original volume in the cut, and there will be practically no subsequent settling of the embankment. This increase in volume is called the **growth of rock.**

Table II shows the approximate number of cubic yards of embankment that can be formed from 1,000 cubic yards of excavation, the number of cubic yards of excavation required for 1,000 cubic yards of embankment, and the per cent. of growth for the various sizes of hard rock.

TABLE II

GROWTH OF ROCK

Character of Material	Embankment Obtained From 1,000 Cubic Yards of Excavation Cubic Yards	Excavation Required for 1,000 Cubic Yards of Embankment Cubic Yards	Growth Per Cent.
Hard rock, large fragments .	1,600	625	60
Hard rock, medium fragments	1,700	587	70
Hard rock, small fragments .	1,800	556	80

EXAMPLE.—How many cubic yards of embankment will be obtained from a rock cut 4,500 cubic yards, if the rock is broken into small fragments?

SOLUTION 1.—Since, from Table II, 1,000 cu. yd. of cut will form 1,800 cu. yd. of fill, the desired number will be

$$\frac{1800}{1000} \times 4{,}500 = 8{,}100 \text{ cu. yd.} \quad \text{Ans.}$$

SOLUTION 2.—Since, from Table II, the growth is 80 per cent., the growth will be 80 per cent. of 4,500 or 3,600; the desired number is, therefore,

$$4{,}500 + 3{,}600 = 8{,}100 \text{ cu. yd.} \quad \text{Ans.}$$

44. The numbers in Table II are necessarily only rough approximations, but the table will give fairly accurate results if the rock is hard rock. If the rock is of a soft earthy nature, or is what is known as *rotten rock*, the percentage of enlargement after excavation is less, and there is more or less subsequent shrinkage. The transition from hard rock to soft earth is so gradual that the engineer must estimate to the best of his ability between the extremes of a 60- to 80-per-cent. growth in an embankment formed from solid

rock to a 10-per-cent. or possibly a 25-per-cent. shrinkage in the case of very soft vegetable soil. Experience alone will determine what value should be chosen within this large range of extreme values. It should be kept in mind that a very soft vegetable soil or a material resembling quicksand is not suitable as a material for embankments, and, therefore, when such material is excavated, it is often better to discard it, and, if necessary, to borrow material with which to form adjacent embankments.

On account of these uncertainties in shrinkage and growth, it is the invariable custom to compute all earthwork and rockwork by measuring the volume of the original excavation, whether it is a cut or a borrow pit. The contractor is then required to dispose of all excavated material where directed, subject to an allowance for haul, as described later, and he need not concern himself with the question of how much embankment will be made from the excavated material.

EXAMPLES FOR PRACTICE

1. Find the amount of embankment that can be formed from a cut of 3,200 cubic yards in loam. Ans. 2,816 cu. yd.

2. Find the amount of embankment that can be formed from a cut of 2,500 cubic yards in hard rock, if the rock is broken into medium-sized fragments. Ans. 4,250 cu. yd.

3. Find the volume required for a borrow pit in sandy soil to furnish 3,590 cubic yards of embankment. Ans. 3,902 cu. yd.

ACCURACY OF EARTHWORK COMPUTATIONS

45. The volumes obtained for earthwork are never more than approximations to the true volumes, and their error frequently amounts to several cubic yards in a prismoid 100 feet long. In cuts, this error is principally caused by the unevenness of the natural surface of the ground between successive cross-sections. The volume computed is that of a prismoid, which is a perfectly regular geometrical solid having straight edges, and the area and shape of any section of which varies uniformly from one base to the other. Evidently, the rough surface of the natural soil never

exactly coincides with one face of any such prismoid; a slight mound or hollow may cause the actual volume to differ by several cubic yards from the computed volume. The accuracy of the computed volumes could be increased by taking cross-sections at very frequent intervals; but, as explained in Art. **20**, the cost of this additional labor usually prevents it. In fills, there is not only the same error arising from the irregularity of the natural surface on which the embankment rests, but there is also a much greater error arising from the uncertainty of the amount of shrinkage that the material of the embankment will undergo.

Since the results of any computation are uncertain to the extent at the very least of 1 or 2 cubic yards per station, the formulas that have been given in the preceding articles can frequently be applied rapidly in an approximate manner. To take a single example, the expression for C_c in the example of Art. **40** is as follows:

$$C_c = \frac{100}{2 \times 27 \times 637} \times \left(\tfrac{1}{2} \times 16.4 \times 14.9 \times 14 + \tfrac{1}{2} \times 8.6 \times 17.1 \times 17.7\right)$$

Instead of performing in full the operations indicated, the experienced engineer would probably obtain the value of C_c, mentally, somewhat as follows: 14.9×14 is about 200; 17.1×17.7 is about 300; the parenthesis is therefore about $8\tfrac{1}{4} \times 200 + 4\tfrac{1}{3} \times 300 = 2{,}950$ and C_c is $\frac{50}{27} \times \frac{2950}{637} =$ (about) $2 \times 4\tfrac{1}{2} = 9$ cubic yards.

In obtaining the quantity of earth in small volumes also, such as that of the wedge $abcdef$, Fig. 13, or of the pyramid $mnpq$, Fig. 13, an engineer of experience can, by merely glancing at the excavation on the ground, frequently estimate the volumes very closely.

HAULAGE

46. Limit of Free Haul.—The most variable item in the cost of earthwork, and the one that in some cases is the largest single item, is that which depends on the length of haul—that is, the distance through which excavated material must be hauled or transported. Specifications usually require

that a contractor shall deposit excavated material at any place designated by the engineer, and that his bid per cubic yard shall cover the cost of such haul, provided that the distance does not exceed 800 or perhaps 1,000 feet. This extreme distance is called the **limit of free haul.** At the same time it is specified that, if the haul exceeds this distance, there shall be an extra allowance per cubic yard for each distance of 100 feet that the material may be hauled. The sum of all the products obtained by multiplying each cubic yard of earth by the number of stations that it is hauled beyond the specified limit is called the **overhaul.** It is necessary to devise some practicable method of estimating the length of haul, not only for the sake of computing the extra allowance arising from overhaul, but also so that different plans for the disposal of the material may be readily made and compared, and so that the plan involving the least amount of haulage may be readily determined.

47. Computation of Haulage.—It is evidently impracticable to compute the haul of each individual cartload of earth and to measure its precise volume. Fortunately, this is not necessary. When any given volume of material is to be transported to any given locality, it makes no difference (at least from the theoretical standpoint) how each individual cubic yard of earth may be hauled. The **total haulage** is the sum of all the products obtained by multiplying each volume by the distance through which this volume is hauled. The total haulage is equal to the total number of cubic yards multiplied by the distance between the center of gravity of the original excavation and the center of gravity of the embankment formed by that excavation. This principle makes the computation of haulage comparatively simple.

Fig. 18 shows a profile of the roadbed at the point in which it passes from a cut to a fill. If all the material of the cut is deposited in the position OND, the total haulage will be

$$\text{volume } CMO \times ZZ' = \text{volume } OND \times ZZ',$$

G and G' being, respectively, the centers of gravity of the cut CMO and the embankment ODN.

If the distance CD is less than the limit of free haul, the contractor is entitled to no extra compensation in connection with the above cut. It should be observed also that the contractor is entitled to the benefit of all short hauls; that is, material so moved is not averaged against that which is carried beyond the limit. Therefore, in cuts the material of which is deposited within the specified limit of haul, no computation of haul need be made.

If there is overhaul at any cut, the contractor does not receive extra compensation for that portion of the cut which is not carried beyond the specified limit. Thus, in Fig. 18, if AB is a distance equal to the limit of free haul, and the material AKO is deposited at OLB, it is evident that no extra charge is admissible for this material. But every

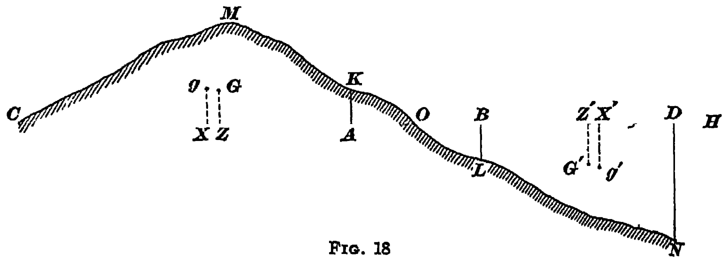

FIG. 18

cubic yard excavated from the left of A and transported to the right of B is overhauled, and, for this, extra compensation must be allowed.

48. The first step in computing the overhaul is to find two points A and B on the profile whose distance apart equals the limit of free haul, and which are so situated that the volume of the cut AKO equals the volume of the fill BOL. This is easily done by trial when the volume of each of the prismoids along CH has been computed. Frequently, it is sufficiently accurate to locate A and B from the profile by shifting these points backwards and forwards until a position is found for which the area KOA is equal to the area BOL.

For moving the material KOA to the position BOL, there is no allowance for overhaul. If g is the center of

gravity of the excavation CKA, and g' that of this material in the embankment $BLND$, and if V is the volume of the cut CKA, then $V \times XX'$ is the haulage of the volume V. But, since this volume must be hauled the length of the free haul AB without extra charge, the overhaul will be

$$\begin{aligned} V \times XX' - V \times AB &= V \times (XX' - AB) \\ &= V \times XA + V \times X'B \end{aligned}$$

The values of $V \times XA$ and $V \times X'B$ are found as follows:

Let v = volume of any prismoid in cut;
a = area of its end section nearest to A;
a' = area of its end section most remote from A;
m = distance from A to middle section of prismoid;
l = length of prismoid, in feet;
x = distance from center of gravity of this prismoid to point A.

Then, as may be proved by the use of advanced mathematics,

$$x = m + \frac{l}{6} \times \frac{a' - a}{a' + a}$$

The overhaul of this prismoid from its position in the cut to the point A will therefore be, since overhaul is reckoned in stations,

$$\frac{vx}{100} = \frac{v}{100}\left(m + \frac{l}{6} \times \frac{a' - a}{a' + a}\right)$$

By this formula, the overhaul for each prismoid of the cut is computed for the transportation of this material to the point A. In an exactly similar manner, the overhaul for the transportation of each prismoid to its position in the fill BLN from the point B is found. The sum of the overhauls for all the prismoids of the cut and fill is the desired total overhaul.

If a part of the cut, for example MZO, is hauled in one direction, and the remainder MZC in the other, the overhaul for each part of the cut must be computed separately.

EXAMPLE.—In the example of Art. 28, the point A, Fig. 18, is at Sta. 129, the length of free haul is 600 feet, and the notes showing

the volumes and end areas of the prismoids beyond Sta. 135 are as follows:

Station	End Area	Volume (a)	Volume (b)
137	769	1,422	1,918
136	268	496	2,060
135	844	1,564	
			Sum = 3,978

If 1 cent is paid for each cubic yard hauled one station in the overhaul, find the total allowance for overhaul if the shrinkage of the material in embankment is 10 per cent.

SOLUTION.—The foregoing formula must be applied to each of the prismoids.

1. *For the Cut.*—We have the following tabulation of the end areas and volumes; the end areas are the algebraic sums of one-half the plus and minus areas found in the tabulation of Art. 28, and the volumes are obtained by applying the prismoidal correction to the volumes in column 4 (*b*) of that table.

Station	End Areas	Volume	m	$\frac{l}{6}\left(\frac{a'-a}{a'+a}\right)$	x	$\frac{vx}{100}$
129	368.5	1,195	30	+5	35	418
128 + 40	724.8	883	80	−4	76	671
128	471.9	1,690	150	−1	149	2,518
127	439.4	1,074	250	−7	243	2,610
126	157.4					
		Sum = 4,842				Sum = 6,217

The numbers in the fourth column are the distances from the middle sections of the prismoids to the point *A*, Fig. 18, at Sta. 129, at which point the free haul begins. Thus, the middle section of the prismoid between Sta. 126 and Sta. 127 is at Sta. 126 + 50; the distances from this section to Sta. 129 is (129 − 126.50) × 100 = 250 ft. Similarly, for the prismoid between Sta. 127 and Sta. 128, m = (129 − 127.50) × 100 = 150 ft.

The value of $\frac{l}{6} \times \frac{a'-a}{a'+a}$, for each prismoid, is given in the fifth column. Thus, for the first prismoid,

$$\frac{100}{6} \times \frac{157.4 - 439.4}{157.4 + 439.4} = -7 \text{ ft.}$$

For the second prismoid,

$$\frac{100}{6} \times \frac{439.4 - 471.9}{439.4 + 471.9} = -1 \text{ ft.}$$

and similarly for the others.

The numbers in the sixth column are the sums of the corresponding numbers in the fourth and fifth columns; each of these numbers in the sixth column is the distance from the point A, Fig. 18, to the center of gravity of the corresponding prismoid.

Finally, the overhaul for each prismoid is the product of the volume in the third column by the distance x in the sixth column. These products are written in the seventh column; but, since the distance x is expressed in feet, and the allowance is 1 cent per cubic yard per station, each product is divided by 100 before writing it in the seventh column. The sum of the numbers in the seventh column is 6,217; the overhaul for the cut is therefore the equivalent of 6,217 cu. yd. overhauled one station.

2. *For the Fill.*—The total volume of the cut is 4,842 cu. yd. Since the shrinkage is 10 per cent., the volume of this material when placed in the embankment will be 4,842 − 484 = 4,358 cu. yd. Since the volume of the embankment between Sta. 135 and Sta. 137 is 4,378 cu. yd., the embankment made from the cut practically ends at Sta. 137. Therefore, the point D, Fig. 18, may be taken as Sta. 137.

The computation of overhaul for fill between Sta. 135, or B, Fig. 18, and the center of gravity of each prismoid is now computed exactly as in the case of the cut. The results are shown in the following table:

Station	End Area	Volume	m	$\frac{l}{6} \times \frac{a' - a}{a' + a}$	x	$\frac{v\,x}{100}$
137	769	1,918	150	+ 8	158	3,030
136	268	2,060	50	− 9	41	845
135	844					

Sum = 3,875

The sum of all the values of $\frac{v\,x}{100}$ is 6,217 + 3,875 = 10,092. This is the equivalent of 10,092 cu. yd. overhauled one station. At the rate of 1 cent per cubic yard per station, the allowance for overhaul will be

$$.01 \times 10{,}092 = \$100.92. \quad \text{Ans.}$$

EXAMPLE FOR PRACTICE

If the cut in the following notes extends from Sta. 62 to Sta. 67, and the fill from Sta. 67 on, and if the point A, Fig. 18, is at Sta. 64, find the allowance for overhaul at the rate of 1 cent per cubic yard per station.

The length of free haul is 600 feet, and the allowance for shrinkage is 10 per cent.

Station	End Area	Volume (a)	Volume (b)
72	89	163	773
71	330	610	1,036
70	235	426	
64	323	586	1,104
63	284	518	900
62	210	382	

Ans. $35.03

GRAPHIC COMPUTATIONS

49. The Mass Diagram and the Mass Curve.—The method described in the preceding article for computing the total haulage and the allowance for overhaul is very accurate and can easily be applied when there are but few cuts and fills at which the question of allowance for overhaul needs to be investigated. But in many cases, and especially in *preliminary estimates* of the cost of earthwork, a graphic method is sufficiently accurate. Such a method is briefly outlined in the present and in the following articles.

Fig. 19 represents a **mass diagram**, which consists of the profile $i' n' j' k'$ of the road drawn to the usual vertical and horizontal scales, the subgrade GR, and the **mass curve** ijk, which is drawn above the profile to the same horizontal scale, in a manner to be described presently. Natural conditions frequently determine with practical certainty that no earth will be hauled beyond certain points, as a river crossing. Beginning at some such point, which will be the point i' in Fig. 19, a table like the one given on page 67 is prepared. The first column contains the station numbers, and the second the number of cubic yards of cut or fill between each station and the station immediately preceding. Cut is here indicated by a plus sign, and fill by a minus sign. Even approximate accuracy requires that shrinkage be allowed for;

the third column therefore contains a statement of the character of all material excavated, and in the fourth column is placed the shrinkage factor for that particular class of material. The fifth column contains the volumes obtained from those in the second column by applying the correction for shrinkage. The sixth column is formed by taking the algebraic sum of all the numbers of cubic yards (as given in

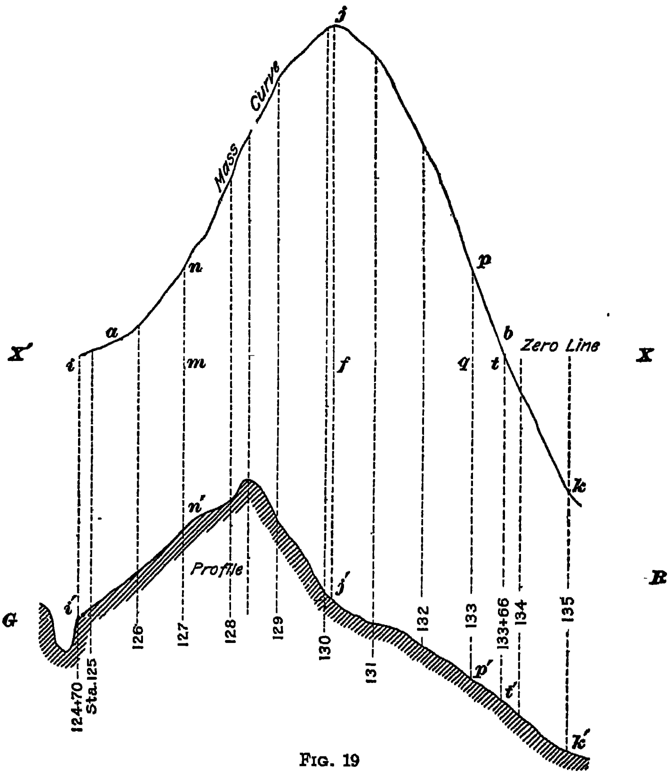

FIG. 19

the fifth column) from the starting point up to the station considered. For example, the number 4,647, given in the sixth column, is the sum of 128, 635, 1,540, and 2,344. The volume − 960, opposite Sta. 131 in the fifth column, means that, between Sta. 130 + 10 and Sta. 131, there was a fill of 960 cubic yards; the number 7,827 in the sixth column is

DETERMINATION OF ORDINATES FOR MASS CURVE

Station	Yards {Cut + / Fill −	Material	Shrinkage Factor Per Cent.	Yards Reduced for Shrinkage	Ordinate to Mass Curve
124 + 70					0
125	+ 142	Gravel and clay	− 10	+ 128	+ 128
126	+ 706	Gravel and clay	− 10	+ 635	+ 763
127	+ 1,711	Gravel and clay	− 10	+ 1,540	+ 2,303
128	+ 2,605	Gravel and clay	− 10	+ 2,344	+ 4,647
+ 40	+ 1,280	Gravel and clay	− 10	+ 1,152	+ 5,799
129	+ 1,688	Gravel and clay	− 10	+ 1,519	+ 7,318
130	+ 1,422	Gravel and clay	− 10	+ 1,280	+ 8,598
+ 10	+ 210	Gravel and clay	− 10	+ 189	+ 8,787
131	− 960			− 960	+ 7,827
132	− 2,340			− 2,340	+ 5,487
133	− 3,285			− 3,285	+ 2,202
134	− 3,328			− 3,328	− 1,126
135	− 2,488			− 2,488	− 3,614

obtained by subtracting from the previous sum total 8,787 the number 960. To obtain the next number 5,487, 2,340 is subtracted from 7,827. When Sta. 134 is reached, these successive subtractions produce a negative result.

50. The **mass curve** ijk is constructed by laying off as abscissas the distances from the starting point, in this case i', and as ordinates the corresponding values in the sixth column of the table. The axis of abscissas $X'X$, called the **zero line**, is taken at any convenient distance above the profile; and the ordinates of the curve are obtained by producing upwards the vertical lines of the profile. Thus, for Sta. 127, the vertical line through that station in the profile is produced, and on it is laid off, above $X'X$, the distance mn to represent, to any convenient scale, the number 2,303 found in the sixth column of the table horizontally opposite Sta. 127. The ordinates corresponding to negative numbers in the sixth column are laid off below the zero line. The vertical scale chosen depends on the magnitude of the earthwork. It should be as large as practicable, since the accuracy of the results is greater the greater the scale used. For a light cutting, a scale of 1,000 cubic yards per inch may not make the diagram too large; but, as the work becomes heavy, 5,000 or even 10,000 cubic yards per inch might be allowable. The scale chosen in Fig. 19 is 5,000 cubic yards per inch.

51. Properties of the Mass Curve.—By comparing Fig. 19 with the notes from which it was constructed, it will be observed that the ordinate corresponding to any station represents the amount of material that can be utilized for embankment beyond that station. Thus, the ordinate mn, corresponding to Sta. 127, represents the total amount of material that can be used for filling anywhere beyond Sta. 127; in this case, this is the same as the total amount excavated up to Sta. 127, since the road up to that station is all in cut, and no part of the excavated material has been used. The ordinates therefore increase up to a station where some of the material excavated begins to be used for embankments;

that is, up to a point where the road changes from cut to fill. That station is j' in Fig. 19; the corresponding ordinate of the curve, which is the maximum ordinate, is fj. This ordinate represents the total amount of material excavated. After passing j', the road is in fill; some of the material excavated is used up, and the ordinates of the mass curve decrease. The ordinate qp, for instance, represents the amount of unused material after the embankment up to p' has been made. The difference between the maximum ordinates fj and qp represents the amount of material used for fill between j' and p'. The mass line crosses the zero line at t, which shows that all the material excavated has been used for fill between j' and t'. Negative ordinates of the curve indicate that more material is required for filling than has been excavated. This material may be obtained from excavation farther along the line.

52. Total Haulage.—The most useful application of the mass curve is to the determination of the total haulage. It can be shown by the use of advanced mathematics that *the total haulage between the stations corresponding to two points where the mass curve crosses the zero line is numerically equal to the area included between the mass curve and the zero line.* Thus, in Fig. 19, the total haulage between i' and t' is numerically equal to the area included between the curve ijt and the zero line it. The **average haulage** is obtained by dividing the total haulage by the total volume of excavated material.

The area of the mass curve can be determined by any of the methods explained in *Plane Trigonometry*, Part 2. Care should be taken to make the proper reduction according to the method and scales used. Haulage is usually expressed by a number of *cubic yards* transported a distance of so many *stations* of 100 *feet*. Therefore, if the area of the curve in square inches is A, and the vertical scale is v cubic yards to the inch, and the horizontal is h stations to the inch, the actual haulage is Ahv. Usually, it is sufficiently accurate to compute the area by treating the portion between two

consecutive ordinates as a trapezoid, using the trapezoidal rule explained in *Plane Trigonometry*, Part 2. In this case, the actual values of the ordinates, as given by the sixth column of the table in Art. **49**, are used, and also the actual values, expressed in stations, of the distances between the ordinates. If this method is followed, no further reduction is necessary.

53. Allowance for Overhaul.—The allowance for overhaul is computed from the mass curve as follows: Draw a line ab, Fig. 19, parallel to the zero line, and whose length is equal to the length of free haul, which in this case will be taken as 800 feet, and such that its ends will be on the mass curve. The position of this line must be determined by trial. In this case, the line ab is at a distance above the zero line which, on the scale of 5,000 cubic yards to the inch, represents 400 cubic yards. This means that only 400 yards are involved. All the material between a and b is hauled a smaller distance than 800 feet, and therefore that material is not involved in the question of overhaul. Even the material to the left of a that is used to make the fill to the right of b is entitled to a haul of 800 feet without extra charge. Extra charge is based on the *excess* distance that these 400 cubic yards must be hauled. This excess haulage is, therefore, measured by the sum of the two triangles in the corners of the mass curve, one of them at the left of a and the other at the right of b. In this case, the allowance is evidently so small that it might be ignored, but for the sake of illustration it will be worked out. The point a corresponds to Sta. 125 + 60. The base of the triangle between ia and the zero line is, therefore, 90 feet, or .90 station, and the area is $\frac{1}{2} \times .90 \times 400 = 180$. The point b comes at Sta. 133 + 55, and the area of the triangle between bt and the zero line is, therefore, $\frac{1}{2} \times .11 \times 400 = 22$. Adding 22 to 180 we have 202. An allowance is sometimes made of 1 cent for each cubic yard hauled each 100 feet of excess distance. On this basis, the allowance in this case would be $2.02, since the sum of the two areas represents a

haulage of 202 cubic yards hauled an excess distance of one station—or, to put it more correctly, 400 yards hauled an average excess distance of .51 station or 51 feet.

COST OF EARTHWORK

54. General Considerations.—The cost of earthwork is a very variable quantity, which depends on many different items of cost. A reliable estimate of it can be made only by one that is enabled to study the local conditions and whose judgment has been trained by experience in such work. The only safe rule for one of limited experience is to classify all the items and analyze them carefully. The principal items whose cost he will ascertain as closely as possible are those given in the following article.

55. Items of Cost.—The first general item is that of **loosening** material. By hand methods, this is accomplished with picks, plows, steam shovels, or, if the material is very hard, by blasting. The steam-shovel method includes the item of loading, which is made a separate item in hand work. The blasting method is so important that it will be treated in a future article.

The item of **loading** will be a separate item when the material is loaded by shovelers, or shoveled by hand into carts or cars. When rock has been blasted out and the rock is of such character that it becomes loosened in rather large masses, it may require to be loaded with derricks. This requires a separate item of cost.

The item of **hauling** is perhaps the most variable item in the cost of earthwork. Although there are some parts of this expense that are independent of the distance, yet the cost of hauling depends very largely on the length of the haul, and the magnitude of this item justifies the elaborate calculations for haul that have been referred to in previous articles. The judgment of the engineer is called on to decide the best method of haul for given conditions. As an extreme limit of short haul may be mentioned side-hill work, where the material excavated from the up-hill side is

formed into an embankment on the down-hill side, and the labor of haulage is reduced to the throwing of the earth a distance of 6 or 8 feet. Scrapers are utilized for very short distances, and in this case the total cost of operating the scraper covers the items of loading and hauling. In very soft material, it may even include the item of loosening, although it will usually pay to have the material loosened with a plow before using the scrapers. Wheelbarrows are also employed for short distances. As the distance increases, economy requires that wheelbarrows and drag scrapers be discarded, although wheeled scrapers may be economically used for distances up to 300 or 400 feet. Carts and horses are economically used for distances up to 1,000 feet. If the distance is very much greater, and especially if the magnitude of the work is very great, it becomes economical to lay a temporary track and use cars hauled by horses or mules. If the magnitude of the work justifies a still more extensive plant, the track is made heavier, the cars have a larger capacity, and are hauled by locomotives. The magnitude of the work may increase until the track, cars, and locomotives are all standard railroad size, and the haul may be economically made for a distance of 5 or even 10 miles. Only the best judgment aided by experience will decide when it is the most economical to haul earthwork for a long distance or to borrow and waste material when the cuts and fills are not evenly balanced.

56. Some minor items that form a very small percentage of the total cost are here mentioned, chiefly to call attention to the fact that they are proper items of expense, and that the neglect to incur these expenses may result in a far greater loss than any supposed economy by avoiding them. One such item is **spreading,** which refers to the expense of keeping a few laborers employed in properly disposing and compacting the individual loads deposited by carts and cars. Another item is that of **keeping the roadways in order.** In the case of cars running on rails, this item is considerable, and is of course absolutely essential,

but a little labor in this respect, even when carts and horses or wheelbarrows are used, is true economy.

An estimate of the items of **repairs, wear, depreciation,** and **interest on cost of plant** will never be neglected by an experienced contractor, especially as these items are very large in earthwork operations. Shovels and picks wear out in a short time. Carts and cars require constant expense for repairs and maintenance, and their maximum life is but a few years at the most. The items of **superintendence** and **incidentals** should not be neglected.

BLASTING

57. Explosives.—The cost of blasting depends principally on: (1) *cost of explosives;* (2) *cost of drilling;* and (3) *cost of exploding a charge.* The various blasting compounds now used vary from ordinary blasting powder to the higher grades of explosives, of which nitroglycerine is the most common basis. Pure nitroglycerine is very seldom used, on account of the difficulty of handling it safely. The explosive compounds commonly used consist generally of a variable percentage of nitroglycerine mixed with some explosive material, which is not only of less cost but makes the compound less liable to explode prematurely. Dynamite is made by saturating inexplosive material with varying proportions of nitroglycerine. The inert matrix is capable of absorbing about 75 per cent. of nitroglycerine; in other words, 1 pound of that grade of dynamite would contain about .75 pound of nitroglycerine. This is called No. 1 dynamite. Such dynamite is about six times as powerful as the same weight of black blasting powder. The blasting powder is usually spoken of as slow burning, but its effect is greater in a soft tough rock than the nitroglycerine compounds, which are detonating, and are most effective in shattering a brittle rock. Some powders are called *slow burning* because it has been proved that the fire is communicated from grain to grain. A *detonating powder* is one that, when jarred, or detonated, undergoes a chemical change

that affects all parts simultaneously and instantly transforms the solid particles into a very hot high-pressure gas. The expansion of this gas causes the explosion.

It is often said that dynamite can be set on fire and will burn harmlessly without exploding. Since numerous explosions have occurred, even when heating dynamite to thaw it out, it is probable that a chemical change may take place in dynamite that causes it to become more explosive than when in its normally pure state. Fortunately, other explosives are now being introduced that are far less dangerous than dynamite, since it seems to be impossible to explode them except by the normal process, using a fulminating cap, as explained later.

58. Drilling.—Hand drilling may be either *churn drilling* or *hammer drilling*. In any kind of work where the drill holes may be made vertical, the churn drill is the most economical hand method. The work of operating a churn drill somewhat resembles the process of churning milk. A rod or pipe about 6 to 8 feet long is welded to a steel drill. The drill is raised by hand and allowed to fall by its own weight. As it is raised, it is slightly rotated so that the edge drops in a different direction for each fall. Where the space is confined, as in a tunnel heading, or where the nature of the rock strata requires that the drill holes shall be inclined, or perhaps horizontal, hand drilling must be done by hammer. By the light-hammer method, one man handles both the drill and the hammer. It is found that the light-hammer method is more expeditious, although perhaps not so economical as the heavy-hammer method. By the latter method, one man holds the drill and two or more men use heavy sledge hammers and strike the drill in turn, the drill being slightly rotated between each two strokes. Machine drilling is far cheaper than hand drilling, provided the magnitude of the work is sufficient to justify the installation of such a plant. The best form of drill is that which has a plain chisel edge, slightly wider than the diameter of the rod and slightly rounded along the edge.

59. Tamping is a very necessary feature of successful blasting. The tamping material is usually the earth or powdered rock that is at hand, but clay is far better and should be used if readily obtainable. The tamping should be done with a *wooden* or *copper* bar, as an *iron* bar may produce sparks, which will explode the charge.

60. Exploding.—Blasting charges are sometimes exploded by means of a fuse, which is essentially a cord of loosely woven material thoroughly saturated with powder. The fuse is usually wrapped with some protecting material, which, in the better quality of fuse, is made of rubber. Such a fuse is sufficient to explode ordinary blasting powder. Dynamite is exploded by means of the minor explosion of a small easily ignited charge. This charge may be exploded by an ordinary fuse leading into a small charge of powder. A better plan is to use a cap filled with fulminate of mercury. This is a powerful and costly explosive, but the quantity required in each cap is very small. The caps are exploded by electricity. In one form they are provided with a platinum wire, which is heated to redness by the passage of an electric current. Such work is almost invariably done by preparing a very large number of holes and wiring them to some central point from which they are all exploded, either simultaneously or in quick succession. To explode them simultaneously is simpler and takes less wiring, but there is the disadvantage that it is uncertain whether each blast has exploded.

61. Cost.—Roughly speaking, the cost of blasting varies between the extremes of 30 cents per cubic yard for brittle rock to about $1 per cubic yard for rock that is hard and tough, and in which the strata are so inconveniently placed that it is difficult so to drill the holes that the blast will have its greatest efficiency.

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NOTE.—In this volume, each Section is complete in itself and has a number. This number is printed at the top of every page of the Section in the headline opposite the page number, and to distinguish the Section number from the page number, the Section number is preceded by a section mark (§). In order to find a reference, glance along the inside edges of the headlines until the desired section number is found, then along the page numbers of that Section until the desired page is found. Thus, to find the reference "Acute angles, §13, p5," turn to the Section marked §13, then to page 5 of that Section.

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